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GEOIDS AS TWO DIMENSIONAL HYPERSURFACES

Abstract

We propose a research of geoids as real hypersufaces. Moreover, we give adjustment of some classical results on hyperusrfaces to CR submanifolds. Our main focus is to study the properties of second fundamental form which give us information on the shape of a hypersurface.

Keywords: geoids, hypersurfaces, shape operator, CR submanifolds.

ГЕОИД ПОСМАТРАН КАО ДВОДИМЕНЗИОНАЛНА ХИПЕРПОВРШ

Сажетак

У овом раду предлажемо проучавање геоида као реалне хиперповрши. Поред тога, показујемо да неки од класичних резултата на хиперповршима вриједе и на Коши-Римановим подмногострукостима. Изучавамо особине друге фундаменталне форме која нам даје информације о облику хиперповрши.

Кључне ријечи: геоид, хиперповрши, оператор облика, Коши-Риманове подмногострукости.

1. INTRODUCTION

The Earth's geoid can be explained as the shape that the ocean surface would take under the influence of the gravity of Earth. That is to say the geoid is an imaginary sea level surface. Together with ellipsoid, the geoid determines one of the geophysical actual shape of the Earth. Geoids are mostly studied in gravitational physics. Moreover, they are used for GPS satellites, for studying climate patterns and related fields. It is interesting to say that such an important object is studied only in the context of Newtonian gravity so far. In the context of general relativity, the authors in [3] obtained the partial differential equation that the geoid satisfies in their observed model of the Earth. A relatively recent research ([5]) gives new method for studying geoids, called quasi local frames. There we can see a representation of geoid as a two dimensional hypersurface. By far, there are no studies of geoids as purely differential geometry objects.

In this paper we give an idea, an open problem, on what information can we get from geoid studied as a hypersurface in an ambient manifold using differential geometry methods. We would like to know what properties geoid hypersurface has, or does it belong to one of the familiar classes of hypersurfaces, such as hypersurfaces from Takagi classification ([7]). Problems like finding principal curvatures, parallelism of the shape operator of the hypersurface normal are of great importance to solve.

We will show one of the famous results on real hypersurfaces in complex space forms and our original result, which gives a generalization to CR submanifolds. In our settings, an ambient manifold is complex space form, i.e. a Kähler manifold of constant holomorphic sectional curvature. The most important examples of complex space forms are complex Euclidean space, complex projective space and complex hyperbolic space.

Let $\bar{M}$ be an $(n+p)$-dimensional complex space form, i.e. a Kähler manifold of constant holomorphic sectional curvature $4c$, endowed with metric g. Let M be an n-dimensional real submanifold of $\bar{M}$ and J be the complex structure of $\bar{M}$. For a tangent space $T_x(M)$ of M at x, we put $H_x(M) = JT_x(M) \cap T_x(M)$. Then, $H_x(M)$ is the maximal complex subspace of $T_x(M)$ and is called the holomorphic tangent space to M at x. If the complex dimension $dim_{\mathbb{C}} H_x(M)$ is constant over M, M is called a Cauchy-Riemann submanifold or briefly a CR submanifold and the constant $dim_{\mathbb{C}} H_x(M)$ is called the CR dimension of M. If, for any $x \in M$, $H_x(M)$ satisfies $dim_{\mathbb{C}} H_x(M) = \frac{n-1}{2}$ then M is called a CR submanifold of maximal CR dimension. It follows that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus span\xi_x$, for any $x \in M$.

A real hypersurface is a typical example of a CR submanifold of maximal CR dimension. The study of real hypersurfaces in complex space forms is a classical topic in differential geometry and the generalization of some results which are valid for real hypersurfaces to CR submanifolds of maximal CR dimension may be expected. For instance, nonexistence of real hypersurfaces with the parallel shape operator and real hypersurfaces with the second fundamental form satisfying $h(JX,Y) - Jh(X,Y) = 0$, in nonflat complex space forms, is proven.

In this paper we study the conditions that the shape operator of the distinguished vector field ξ is parallel on CR submanifolds of maximal CR dimension in complex space forms.

Our paper is organized as follows. In the first part we give some basic definitions and properties of hypersurfaces and geoids. After that, a generalization of the famous result of R. Niebergall and P.J. Ryan ([4]) is given.

2. PRELIMINARIES

2.1. GEOIDS

The gravity on the Earth is defined as a resultant force of universal gravitational attraction and Earth's rotation;

$$\mathrm{grad} W = \mathrm{grad}\, U + \mathrm{grad}\ \Phi,$$

where W is the potential function of gravity, U is the potential of universal gravitation and Φ is the potential of the rotational force of the Earth.

Definition 2.1. For a function $\mathrm{f}: \mathrm{U} \subset \mathbb{R}^3 \to \mathbb{R}$ the level surface of value c is the surface S in U on which $\mathrm{f} = \mathrm{c}$.

Equipotentials are surfaces of constant gravitational potential. The Earth's gravity potential field contains infinity many level surfaces, which are parallel to each other. The geoid is one of those surfaces with a special potential value. Let U_0 be a potential of reference ellipsoid, of which level surface well approximate the mean sealevel (a “tentative geoid”). Then geoid is defined as the level surface $W = U_0$ (See Figure 1).

Usually, the potential function of gravity is calculated by considering Dirichlet problem with regarding geoid locally as a sphere. Then the potential is given as a surface spherical harmonics with undetermined coefficients. To determine the coefficients, we need surveys of position and gravity anomaly.

Satellite orbit analysis and steady gravity survey enable us to map the geoid accurately ([6]) (See Figure 2). You can read more on mathematical geodesy in [2].

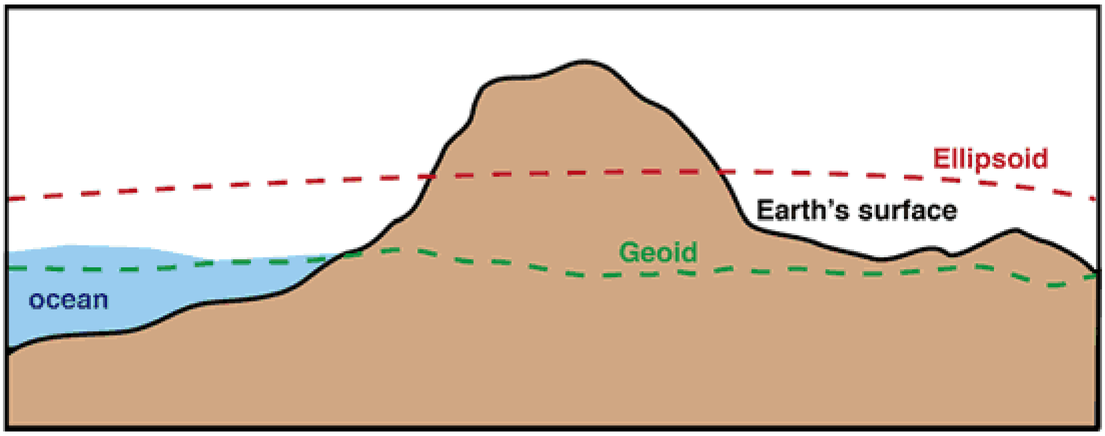

Figure 1. *Contrast of the Geoid model with an Ellipsoid and cross-section of the Earth's surface (from a webpage of US government: https://www.usgs.gov/)*

Figure 2. *Global Geographical Mapping of Geoid (from the webpage of International Centre for Global Earth Model: \http://icgem.gfz-potsdam.de/home)*

2.2. REAL HYPERSURFACES AND CR SUBMANIFOLDS

Let $\overline{M}$ be an $(n+p)$-dimensional complex space form with Kähler structure $(J, \overline{g})$ and of constant holomorphic sectional curvature $4c$. Let M be an n-dimensional CR submanifold of maximal CR dimension in $\overline{M}$ and $\iota: M \to \overline{M}$ an immersion. Also, we denote by ι the differential of the immersion. The Riemannian metric g of M is induced from the Riemannian metric $\overline{g}$ of $\overline{M}$ in such a way that $g(X, Y) = \overline{g}(\iota X, \iota Y)$, where $X, Y \in T(M)$. We denote by $T(M)$ and $T^{\perp}(M)$ the tangent bundle and the normal bundle of M, respectively.

On M we have the following decomposition into tangential and normal components:

$$J\iota X = \iota FX + u(X)\xi,\ X \in T(M). \tag{1}$$

Here F is a skew-symmetric endomorphism acting on $T(M)$ and $u: T(M) \to T^{\perp}(M)$.
Since $T_1^{\perp}(M) = \{\eta \in T^{\perp}(M) | \overline{g}(\eta, \xi) = 0\}$ is J-invariant, from now on we will denote the orthonormal basis of $T^{\perp}(M)$ by $\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = \frac{p-1}{2}$. Also, $J\xi$ is the vector field tangent to M and we write

$$J\xi = -\iota U. \tag{2}$$

Furthermore, using (1), (2) and the Hermitian property of J implies

$$F^2X = -X + u(X)U, \tag{3}$$

$$FU = 0, \tag{4}$$

$$g(X, U) = u(X). \tag{5}$$

Next, we denote by ∇ and $\overline{\nabla}$ the Riemannian connection of M and $\overline{M}$, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. They are related by the following Gauss equation

$$\overline{\nabla}_{\iota X}\iota Y = \iota\nabla_X Y + h(X, Y), \tag{6}$$

where h denotes the second fundamental form, and by Weingarten equations

$$\overline{\nabla}_{\iota X}\xi = -\iota AX + D_X\xi = -\iota AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}, \tag{7}$$

$$\overline{\nabla}_{\iota X}\xi_a = -\iota A_aX + D_X\xi_a = -\iota A_aX - s_a(X)\xi + \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\}, \tag{8}$$

$$\overline{\nabla}_{\iota X}\xi_{a^*} = -\iota A_{a^*}X + D_X\xi_{a^*} = -\iota A_{a^*}X - s_{a^*}(X)\xi + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\}, \tag{9}$$

where the s's are the coefficients of the normal connection D and $A, A_a, A_{a^*}; a = 1, \dots, q$, are the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*} , respectively. They are related to the second fundamental form by

$$h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^{q}\{g(A_aX, Y)\xi_a + g(A_{a^*}X, Y)\xi_{a^*}\}. \tag{10}$$

Since the ambient manifold is a Kähler manifold, using (1), (2), (8) and (9), it follows that

$$A_{a^*}X = FA_aX - s_a(X)U, \tag{11}$$

$$A_aX = -FA_{a^*}X + s_{a^*}(X)U, \tag{12}$$

$$s_{a^*}(X) = u(A_aX), \tag{13}$$

$$s_a(X) = -u(A_{a^*}X), \tag{14}$$

for all X, Y tangent to M and $a = 1, \dots, q$.

Moreover, since F is skew-symmetric and A_a and $A_{a^*}; a = 1, \dots, q$, are symmetric, (11) and (12) imply

$$g\big((A_aF + FA_a)X, Y\big) = u(Y)s_a(X) - u(X)s_a(Y), \tag{15}$$

$$g\big((A_{a^*}F + FA_{a^*})X, Y\big) = u(Y)s_{a^*}(X) - u(X)s_{a^*}(Y), \tag{16}$$

for all $a = 1, \dots, q$. Finally, the Codazzi equation for the distinguished vector field ξ becomes of the following form

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{u(X)FY - u(Y)FX - 2g(FX,Y)U\} + \sum_{a=1}^{q}\{s_a(X)A_a Y - s_a(Y)A_a X\}$$
$$+ \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*} Y - s_{a^*}(Y)A_{a^*}X\} \quad (17)$$

for all X, Y tangent to M.

3. CR SUBMANIFOLDS WITH PARALLEL SHAPE OPERATOR

Here, we will give one well known result about hypersurfaces with the parallel shape operator.
Theorem 3.1. Let M be an n-dimensional, where n ≥ 3, hypersurface in a complex space form of constant holomorphic sectional curvature 4c $\neq 0$. Then the shape operator A of M cannot be parallel.
We will study the same condition on CR submanifolds of maximal CR dimension in complex space forms. Therefore, we have the next two theorems.
Theorem 3.2. Let M be an n-dimensional CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\mathrm{M}}, \mathrm{J}, \overline{\mathrm{g}})$, where n ≥ 3 and the constant holomorphic sectional curvature of $\overline{\mathrm{M}}$ equals 4c. Let the distinguished vector field ξ be parallel with respect to the normal connection D and A be the shape operator of ξ. If ∇A $= 0$ on M, then $\overline{\mathrm{M}}$ is a Euclidean space.
Proof. Putting $Y = U$ in Codazzi equation (17), we get

$$(\nabla_X A)Y - (\nabla_Y A)X = -cFX + \sum_{a=1}^{q}\{s_a(X)A_a U - s_a(U)A_a X\} + \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*} U - s_{a^*}(U)A_{a^*}X\}.$$

From the assumption of the theorem and the last equation, we get

$$cFX = 0,$$

from which we conclude that $c = 0$. □

Theorem 3.3. Let M be an n-dimensional CR submanifold of maximal CR dimension in an $(\mathrm{n}+\mathrm{p})$-dimensional complex space form $(\overline{\mathrm{M}}, \mathrm{J}, \overline{\mathrm{g}})$, where n ≥ 3 and the constant holomorphic sectional curvature of $\overline{\mathrm{M}}$ equals 4c. Let p $<$ n and A be the shape operator of the distinguished vector field ξ. If ∇A $= 0$ on M, then $\overline{\mathrm{M}}$ is a Euclidean space.
Proof. After putting $Y = U$ in (17) and using the assumption of the theorem, we get

$$-cFX + \sum_{a=1}^{q}\{s_a(X)A_a U - s_a(U)A_a X\} + \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*} U s_{a^*}(U)A_{a^*}X\} = 0. \quad (18)$$

After multiplying the equation (18) by an arbitrary $Y \in T(M)$, we get

$$-cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) - s_a(U)g(A_aX,Y)\}$$
$$+ \sum_{a=1}^{q}\{s_{a^*}(X)g(A_{a^*}U,Y) - s_{a^*}(U)g(A_{a^*}X,Y)\} = 0. \quad (19)$$

Interchanging X and Y in (19) and subtracting (19) and the resulting equation, we get

$$-2cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) + s_{a^*}(U)g(A_{a^*}U,Y)\}$$
$$+ \sum_{a=1}^{q}\{s_a(Y)g(A_aU,X) + s_{a^*}(Y)g(A_{a^*}U,X)\} = 0. \quad (20)$$

Now, using (5), (13) and (14), from the last equation it follows that

$$cFX = \sum_{a=1}^{q} \{s_{a^*}(X) A_{a^*} U + s_a(X) A_a U\}. \tag{21}$$

On the other hand, if we put

$$0 = \sum_{a=1}^{q} \{c_{a^*}(X) A_{a^*} U + s_a(X) A_a U\}, \tag{22}$$

where c_{a^*} and c_a are constants; $a = 1, \dots, q$, by scalar multiplication of (22) with an arbitrary $X \in T(M)$ using $\overline{g}(\iota A_a X, \iota Y) = \overline{g}(h(X,Y), \xi_a)$, $\overline{g}(\iota A_{a^*} X, \iota Y) = \overline{g}(h(X,Y), \xi_{a^*})$; $a = 1, \dots, q$, and (6), it follows that

$$0 = \sum_{a=1}^{q} \{c_{a^*} \overline{g}(\overline{\nabla}_{\iota U} X, \xi_{a^*}) + c_a \overline{g}(\overline{\nabla}_{\iota U} X, \xi_a)\},$$

i.e.

$$0 = \sum_{a=1}^{q} \{c_{a^*} \xi_{a^*} + c_a \xi_a\}.$$

From the last equation and the fact that ξ_{a^*}, ξ_a, $a = 1, \dots, q$, are linearly independent, it follows that $c_{a^*} = c_a = 0$; $a = 1, \dots, q$. Then, we can conclude that $A_{a^*}U$, $A_a U$; $a = 1, \dots, q$, are linearly independent vector fields. It is known that $\operatorname{rank} F = n - 1$ (see [1]), that is why from (21) it follows that there exist a vector field $Y \in T(M)$ such that $Y = FX$ and that Y is orthogonal to the vector fields $A_a U$, $A_{a^*} U$; $a = 1, \dots, q$. Multiplying (21) with $Y = FX$, we get

$$cg(FX, FX) = 0,$$

from which we conclude that $c = 0$. □

LITERATURE:

[1] M. Djorić and M. Okumura, CR submanifolds of complex projective space. Berlin: Springer, 2009.

[2] M. Hotine, Mathematical Geodesy. U.S.: U.S. Government Printing Office, 1969.

[3] S. Kopeikin, E. Mazurova, and A. Karpik, “Towards an exact relativistic theory of Earth’s geoid undulation”, Phys.Lett., vol. A379, pp. 1555-1562, 2015.

[4] R. Niebergall and P.J. Ryan, “Real hypersurfaces in complex space forms”, Tight and Taut Submanifolds, vol. 32, pp. 233-305, 1997.

[5] Marius Oltean et al, “Geoids in general relativity: geoid quasilocal frames”, Class. Quantum Grav., vol. 33, 2016.

[6] E. Sinem Ince et al, “ICGEM - 15 years of successful collection and distribution of global gravitational models, associated services, and future plans”, Earth Syst. Sci. Data, vol. 11, pp. 647-674, 2019.

[7] R. Takagi, “Real hypersurf aces in a complex projective space with constant principal curvatures”, J. Math. Soc. Japan, vol. 27, 1975.