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DEFINITION OF CRACKS FOR THE ULTIMATE DEGENERATION PROCESS OF REVOLVING ELLIPSOIDS

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ABSTRACT

The state of stress around cavities of the revolving ellipsoidal shape (prolate or oblate) in the extreme cases of their degeneration is a very complex problem. Namely, the shape of cavities is defined by an appropriate curvilinear coordinate system, which contains hyperbolic and trigonometric functions. In this paper, the treatment of revolving ellipsoid coordinates has been presented, along with transformations of ellipsoids for corresponding limiting values of their coordinates. Various forms of cracking can be defined through the ellipsoid degeneration process. In addition, the paper is dealing with a definition of the stress state for the limiting cases of degeneration.

Keywords: *revolving ellipsoid, ultimate degeneration process, crack, state of stress*

INTRODUCTION

When considering the stress state around hollow rotating bodies, with an aim to simplify the obtained analytical expressions, systems of curvilinear coordinates that correspond to a given cavity shape are introduced. In addition to the above mentioned reasons, there is another quite important motive, and that is to obtain the components of the stress tensor in perpendicular or tangential directions with respect to a cavity surface. In such a way, a more realistic representation of stresses that are relevant for the analysis is achieved, figure 1,2.

When it comes to cavities of revolving ellipsoidal shape, a corresponding curvilinear coordinate system is introduced in the following manner [1,2]:

Prolate (elongated) ellipsoid

$$x = c\ \mathrm{sh}u\ \sin\varphi\cos\theta;\ \ y = c\ \mathrm{sh}u\ \sin\varphi\sin\theta;\ \ z = c\ \mathrm{ch}u\ \cos\varphi \tag{1}$$

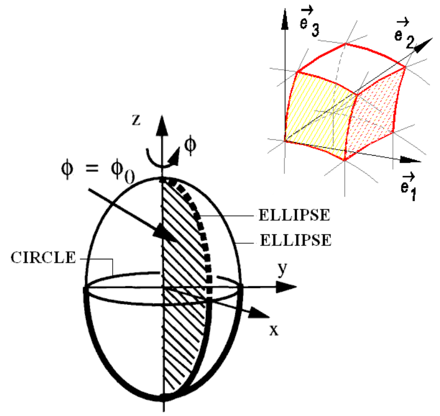

Figure 1. The coordinate system of prolate ellipsoid

Oblate (flattened) ellipsoid

$$x = c\ \mathrm{ch}u\ \sin\varphi \cos\theta;\ \ y = c\ \mathrm{ch}u\ \sin\varphi \sin\theta;\ z = c\ \mathrm{sh}u\ \cos\varphi \tag{2}$$

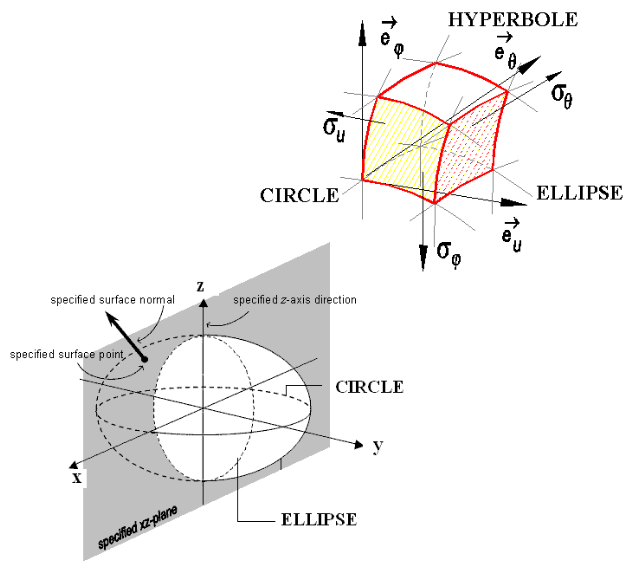

Figure 2. The coordinate system of oblate ellipsoid

where:
$u,\ \varphi,\ \theta$ - elliptical coordinates;
c – adistance between foci.

DEFINITION OF STRESSES AND THE ULTIMATE DEGENERATION PROCESS OF THE PROLATE ELLIPSOID

In order to define cracks, a stress state in the ultimate deformation process of revolving ellipsoids is observed. Starting from the approximate solutions for the stress tensor of the prolate and oblate ellipsoids, two stresses σ_φ and σ_θ have been analysed. Residual stresses and their analysis in the ultimate degeneration process are irrelevant for definition of a crack, and therefore, the expressions for their determination are not presented in this paper. Crack analysis is frequently used in the theory of fractures, as well as the definition of new materials in high technology [3].

The approximate solution for the stresses [1] (for n=2) is introduced in the following form:

$$\begin{aligned}\sigma_\varphi = {} & A_0\left[\varphi_0^1 P_0 Q_0 + \varphi_0^2 P_1 Q_0 + \varphi_0^3 P_0 Q_1\right] + \\ & C_0\left[\varphi_0^4 P_0 Q_1 + \varphi_0^5 P_1 Q_0 + \varphi_0^6 P_1 Q_1\right] + \\ & A_2\left[\varphi_2^1 P_2 Q_2 + \varphi_2^2 P_3 Q_2 + \varphi_2^3 P_2 Q_3\right] + \\ & C_2\left[\varphi_2^4 P_2 Q_3 + \varphi_2^5 P_3 Q_2 + \varphi_2^6 P_3 Q_3\right] + \sigma\end{aligned} \tag{3}$$

$$\begin{aligned}\sigma_\theta = {} & A_0\left[\Theta_0^1 P_0 Q_0 + \Theta_0^2 P_1 Q_0 + \Theta_0^3 P_0 Q_1\right] + \\ & C_0\left[\Theta_0^4 P_0 Q_1 + \Theta_0^5 P_1 Q_0 + \Theta_0^6 P_1 Q_1\right] + \\ & A_2\left[\Theta_2^1 P_2 Q_2 + \Theta_2^2 P_3 Q_2 + \Theta_2^3 P_2 Q_3\right] + \\ & C_2\left[\Theta_2^4 P_2 Q_3 + \Theta_2^5 P_3 Q_2 + \Theta_2^6 P_3 Q_3\right] + \sigma\end{aligned} \tag{4}$$

It can be shown that the constants may be presented as:

$$\begin{aligned} & A_0 = \sigma \cdot \xi_0;\ A_2 = \sigma \cdot \xi_2 \\ & C_0 = \sigma \cdot \eta_0;\ C_2 = \sigma \cdot \eta_2 \end{aligned} \tag{5}$$

where ξ_0, ξ_2, η_0, and η_2 stand for constants.

Introducing the relation (5), the stresses may be written in the following form [1,4,5,6,7]:

$$\begin{aligned}\sigma_\varphi = {} & \sigma(1 + [\frac{1}{h^2}(2 - \frac{1}{\sin^2\varphi})\xi_0 Q_0 + \frac{1}{h^4}\cos^2\varphi[\frac{h^2}{\sin^2\varphi} + c^2)\xi_0 Q_0 - \\ & \frac{c^2}{h^4} chu\,\xi_0 Q_1 + \frac{3}{h^2}(4 - \frac{1}{\sin^2\varphi})\xi_2 P_2 Q_2 + 3\frac{1}{h^4}\cos\varphi[\frac{h^2}{\sin^2\varphi} + c^2)\xi_2 P_3 Q_2 - \\ & 3\frac{c^2}{h^4} chu\,\xi_2 P_2 Q_3 + \frac{c}{h^4} chu[h^2(\alpha - \frac{\cos^2\varphi}{\sin^2\varphi}) - c^2\cos^2\varphi]\eta_0 Q_1 + \frac{c}{h^4}\cos^2\varphi \\ & \times[c^2 ch^2 u - h^2(2-\alpha)]\eta_0 Q_0 + \frac{c}{h^2} chu\cos^2\varphi[(\frac{1}{\sin^2\varphi} - 2\alpha + 2]\eta_0 Q_1 + \\ & \frac{3c}{h^4} chu[h^2(\alpha - \frac{\cos^2\varphi}{\sin^2\varphi}) - c^2\cos^2\varphi]\eta_2 P_2 Q_3 + \\ & \frac{3c}{h^4}\cos\varphi[c^2 ch^2 u - h^2(2-\alpha)]\eta_2 P_3 Q_2 + \\ & \frac{3c}{h^2} chu\cos\varphi[\frac{1}{\sin^2\varphi} - 2\alpha + 4]\eta_2 P_3 Q_3])\end{aligned} \tag{6}$$

$$\begin{aligned}\sigma_\theta = {} & \sigma(1 + [\frac{1}{h^2}(\frac{\cos^2\varphi}{\sin^2\varphi} + \frac{ch^2 u}{sh^2 u})\xi_0 Q_0 - \frac{1}{h^2}\frac{\cos^2\varphi}{\sin^2\varphi}\xi_0 Q_0 - \\ & \frac{1}{h^2}\frac{chu}{sh^2 u}\xi_0 Q_1 + \frac{3}{h^2}[\frac{\cos^2\varphi}{\sin^2\varphi} + \frac{ch^2 u}{sh^2 u}]\xi_2 P_2 Q_2 - \frac{3}{h^2}\frac{\cos\varphi}{\sin^2\varphi}\xi_2 P_3 Q_2 - \\ & \frac{3}{h^2}\frac{chu}{sh^2 u}\xi_2 P_2 Q_3 + \frac{c}{h^2} chu[\frac{\cos^2\varphi}{\sin^2\varphi} + (2-\alpha)]\eta_0 Q_1 + \\ & \frac{c}{h^2}\cos^2\varphi[\frac{ch^2 u}{sh^2 u} - (2-\alpha)]\eta_0 Q_0 - \frac{c}{h^2} chu\cos^2\varphi[\frac{\cos^2\varphi}{\sin^2\varphi} + \frac{ch^2 u}{sh^2 u}]\eta_0 Q_1 + \\ & \frac{3c}{h^2} chu[\frac{\cos^2\varphi}{\sin^2\varphi} + (2-\alpha)]\eta_2 P_2 Q_3 + \frac{3c}{h^2}\cos\varphi[\frac{ch^2 u}{sh^2\varphi} - (2-\alpha)]\eta_2 P_3 Q_2 - \\ & \frac{3c}{h^2} chu\cos\varphi[\frac{\cos^2\varphi}{\sin^2\varphi} + \frac{ch^2 u}{sh^2 u}]\eta_2 P_3 Q_3])\end{aligned} \tag{7}$$

In the above presented relations: $h = \sqrt{sh^2 u + \cos^2 \varphi}$ (8)

$$P_n = P_n(\cos\varphi) = \frac{1}{2^n n!}\frac{d^n}{d(\cos\varphi)^n}(\cos^2\varphi - 1)^n \tag{9}$$

denote Legendre functions of the first kind [6,7],

$$Q_n = Q_n(chu) = \frac{1}{2}P_n(chu)\lg\frac{chu-1}{chu+1} + W_{n-1}(chu) \tag{10}$$

denote Legendre functions of the second kind [6,7], and

$$W_{n-1}(chu) = \frac{2n-1}{1n}P_{n-1}(chu) + \frac{2n-5}{3(n-1)}P_{n-3}(chu) + \ldots \tag{11}$$

For the coordinate system given in (1), two limiting cases of the prolate revolving ellipsoid degeneration differ:

When $\theta = const$, $c = const$, and $u \to 0$, the ellipsoid is degenerated into a *thread of length 2c* [1].
As $\theta \neq const$, $c \to 0$, and $u \to \infty$, the ellipsoid is converted into a *sphere of radius R,* figure 3, where $c\ chu \to c\ shu \to R$.

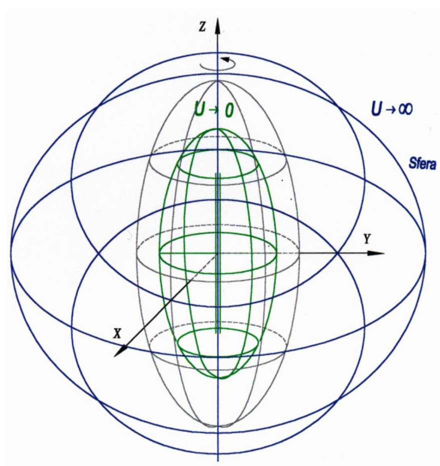

Figure 3. Degeneration of prolate ellipsoid

Considering the two limiting cases of degeneration, the stress values σ_φ and σ_θ will be:

$$\begin{aligned} &\lim_{n\to 0}\sigma_\varphi = \infty; \quad \lim_{n\to\infty}\sigma_\varphi = \sigma \\ &\lim_{n\to 0}\sigma_\theta = \infty; \quad \lim_{n\to\infty}\sigma_\theta = \sigma \end{aligned} \tag{12}$$

The resulting solution is consistent with the solutions reported in the literature for a crack of the ellipsoidal shape, i.e., for the case when the ellipsoid is degenerated into a thin filament of length 2*c* (the *Griffith's crack*).

In order to obtain a solution for a crack of a circular-disk-shape, which is also reported in the literature, it is necessary to point out the following facts:

With respect to the expression for the stress $\sigma_{\theta,}$ as φ approaches $\pi/2$, we have the following stress equation:

$$\sigma_\theta = \sigma(1+[\frac{1}{c^2 sh^2 u}\xi_0 Q_0 - \frac{1}{c^2 chush^2 u}\xi_0 Q_1 - \frac{3}{2c^2 sh^2 u}\xi_2 Q_2 + \frac{3}{c^2 chush^2 u}\xi_2 Q_3 - \frac{1}{cchu}(2-\alpha)\eta_0 Q_1 - \frac{3}{2cchu}(2-\alpha)\eta_2 Q_3]) \tag{13}$$

The stress value (**13)** as $u \to 0$, that is when the circular disk is degenerated to a point, is being infinite ($\sigma_\theta \to \infty$). On the other hand, when along the contour of the circular disk is provided that $u = u_0$, stresses are of final values. This indicates that the solution of a crack should be sought by analysing the limiting cases of degeneration process considering the oblate ellipsoid.

DEFINITION OF STRESSES AND ULTIMATE DEGENERATING PROCESS OF THE OBLATE ELLIPSOID

Expressions for determining the stresses σ_φ and σ_θ are obtained from the solution for the oblate ellipsoid and are given in the following form [3]:

$$\sigma_\varphi = \sigma\frac{\omega}{2\Omega}[2(1+\nu)\omega q^2 - q(2\nu\omega + 2\nu + 7\omega) + 1 + 4\omega + 2\nu] \tag{14}$$

$$\sigma_\theta = \sigma\frac{1}{2\Omega}(\omega[2(1+\nu)\omega q^2 - q[\omega + 6 + 4\nu(2\omega - 1)] + 4\nu\omega + 3] + 2(1-\nu)) \tag{15}$$

where:

$$\begin{aligned}
&\Omega = -(1+\nu)q^2\omega^2 + q[\omega^2 - 2(1-\nu)\omega] + \omega + 1 - \nu \\
&q = \frac{1}{\sqrt{\omega - 1}}\operatorname{arctg}\sqrt{1-\omega} && \text{za} \quad \omega > 1 \\
&q = \frac{1}{\sqrt{\omega - 1}}[\ln(1+\sqrt{1-\omega}) - \frac{1}{2}\ln\omega] && \text{za} \quad \omega < 1 \\
&\omega = \frac{ch^2 u}{sh^2 u}
\end{aligned} \tag{16}$$

$\cos\varphi = 0$ (circular disc of radius *chu*).
As $\theta = const$, $\varphi \to \pi/2$ *const,* and $u \to 0$, the ellipsoid is converted into a *circular disc,* figure 4.

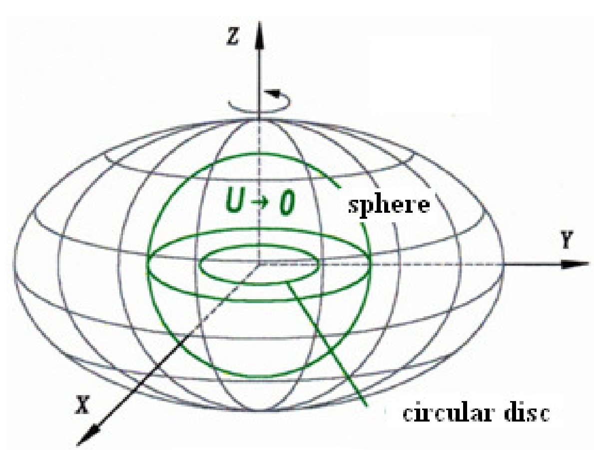

Figure 4. Degeneration of oblate ellipsoid

By analysing the obtained expressions for the stresses and under the conditions $u \rightarrow 0 \Rightarrow shu \rightarrow 0 \Rightarrow$ $z = shu \cos\varphi \rightarrow 0$, the circular disc is being reduced to a disc of radius r = 1, when the following conclusions arise:

$$\lim_{\substack{n \to 0 \\ \omega \to \infty}} \sigma_\theta = \infty; \qquad \lim_{\substack{n \to \infty \\ \omega \to 1}} \sigma_\theta = \sigma \tag{17}$$

The above solution presents the required solution for a crack of a shape of a circular disc of radius r = 1 (the *Sneddon's crack*).

CONCLUDING REMARKS

In this work, the possibility of obtaining the solution in the field of fracture mechanics for crack analysis has been presented, based on the solutions for the stress state. The analysis of the limiting cases of the revolving ellipsoid degenerating process is very important, in particular when a shape of cracking in a material needs to be determined (whether it is the Griffith's or the Sneddon's crack). Moreover, the presented analysis reveals that the Griffith's crack is to be determined by a stress analysis considering the elongated ellipsoid, whereas the Sneddon's crack is being determined by analysing the flattened ellipsoid. Lastly, the importance of such analysis should be emphasized when rock materials that represent cracked media are under consideration, in which case it is often necessary to determine a shape of a crack.

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REFERENCES

[1] Lukic, D. (1998). Contribution to methods of stress state determination around cavity of rotational ellipsoid shape, by use of elliptic coordinates, PhD thesis, University of Belgrade.
[2] Hobson, E.W. (1931). Spherical and ellipsoidal harmonics, Cambridge.
[3] Sneddon, N., Lowengrub, M. (1969). Crack Problems in the Classical theory of Elasticity, New York.
[4] Neuber, H. (1958). Kerbspannungslehre, Springer-Verlag, Berlin.
[5] Edwards, R.H. (1951). Stress Concentrations Around Spheroidal Inclusions and Cavities, *J. Appl. Mechanics*, Vol.18, pp 19-30.
[6] Lukić, D., Prokić, A., Anagnosti, P. (2009). Stress-strain field around elliptic cavities in elastic continuum, *European Journal of Mechanics - A/Solids*, Vol. 28, No. 1, pp. 86- 93.
[7] Lukić, D., Prokić, A., Brčić, S. (2011). Stress state around cylinrical cavities in isotropic medium, *Archives for Technical Sciences*, Technical Institute of Bijeljina, No **5**, pp. 40-47.