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ON THE MECHANICAL RESPONSE OF A PRESSURIZED FUNCTIONALLY-GRADED CYLINDER

Vlado Lubarda[*]
Department of NanoEgineering and Mechanical and Aerospace Engineering,
University of California, San Diego; La Jolla, CA 92093-0448, USA
Montenegrin Academy of Sciences and Arts, Rista Stijovica 5, 81000 Podgorica, Montenegro

Abstract: A pressurized functionally-graded cylinder is considered made of the material whose elastic moduli vary with the radial distance according to the power-law relation. Some peculiar features of the mechanical response are noted for an incompressible functionally-graded material with the power of radial inhomogeneity equal to two. In particular, it is shown that the maximum shear stress is constant throughout the cylinder, while the displacement changes proportional to 1/r along the radial distance. No displacement takes place at all under equal pressures applied at both boundaries.
Keywords: Elasticity, functionally-graded material, incompressibility, pressurized cylinder, radial inhomogeneity, stress amplification.

1. INTRODUCTION

The effects of radial inhomogeneity of material on the mechanical response in thick-walled cylinders under different loading conditions have been studied by many researchers. The representative references include [1−9]. Unusual characteristics of the response associated with even a small degree of radial nonuniformity have been discussed, such as the stress amplification due to enhanced material stiffness, or the stress shielding due to softening of material near a hole. These studies are of importance for the mechanics of composite materials, processing of the functionally-graded materials, wood industry, etc.

In this paper we consider a pressurized functionally-graded cylinder made of the material whose elastic moduli vary with the radial distance according to the power-law relation. The mechanical response is evaluated for compressible and incompressible functionally-graded materials. Some distinct features of stress response are noted in the case of the material with the quadratic-type radial inhomogeneity. It is shown that in this case the maximum shear stress is constant throughout an incompressible cylinder, while the displacement changes proportional to 1/r along the radial distance. The displacement in the cylinder vanishes identically under equal pressures applied at both boundaries. The presented analysis can be extended to other types of boundary conditions, which correspond to prescribed displacements at both boundaries, or mixed boundary conditions, along the lines presented in [10].

2. FUNCTIONALLY-GRADED CYLINDER

We consider a long hollow cylinder under plane strain conditions, made of isotropic material which is functionally-graded (nonuniform) in the radial direction so that its elastic shear modulus varies according to the power-law relation

$$\mu(r) = \mu^b \left(\frac{r}{b}\right)^m . \tag{1}$$

The exponent m is a positive or a negative real number, reflecting the degree of nonuniformity of the material, and μ^b is the shear modulus at the outer boundary $r = b$. If $m > 0$, the elastic stiffness increases outward from the inner ($r = a$) to the outer ($r = b$) radius of the cylinder, while it decreases for $m < 0$. The shear moduli at the two boundaries are related by $\mu^a = c^m \mu^b$, where $c = a/b$ is the aspect ratio of the hollow cylinder. Generally speaking, for $m > 0$ there is a tendency for stress shielding, and for $m < 0$ for stress amplification. Other types of nonuniformity can be considered, such as the exponential variation [5]. The stress-strain relations for infinitesimally small elastic deformations are

[*] Corresponding author: vlubarda@ucsd.edu

$$\varepsilon_r = \frac{1}{2\mu}\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right],$$

$$\varepsilon_\theta = \frac{1}{2\mu}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]. \quad (2)$$

The coefficient of lateral contraction ν is assumed to be independent of r, which simplifies the mathematical aspects of the analysis [7−9]. As a consequence, the Young's modulus of elasticity also varies according to the power-law relation [1,3,9]

$$E(r) = E^b\left(\frac{r}{b}\right)^m, \; E^b = 2(1+\nu)\mu^b. \quad (3)$$

3. GOVERNING DIFFERENTIAL EQUATIONS

If uniform pressure is applied at two boundaries, the circumferential component of displacement is zero, and the radial component u depends on the radial distance r only. Correspondingly, the stress components σ_r and σ_θ are also r-dependent only. In the absence of body force, the equilibrium equation is [11]

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \quad (4)$$

The strain-displacement relations are $\varepsilon_r = du/dr$, and $\varepsilon_\theta = u/r$, with the corresponding Saint-Venant compatibility condition

$$\frac{d\varepsilon_\theta}{dr} - \frac{\varepsilon_r - \varepsilon_\theta}{r} = 0. \quad (5)$$

By the standard procedure it follows that the radial stress is governed by the second-order differential equation

$$r^2\frac{d^2\sigma_r}{dr^2} + (3-m)r\frac{d\sigma_r}{dr} - \frac{1-2\nu}{1-\nu}m\sigma_r = 0. \quad (6)$$

The general solution of this equation is

$$\sigma_r = Ar^{-n_1} + Br^{-n_2}. \quad (7)$$

the integration constants are A and B, and the exponents n_1 and n_2 are

$$n_{1,2} = \frac{1}{2}(2-m\mp s), \; s = \left(4 - \frac{4\nu m}{1-\nu} + m^2\right)^{1/2}. \quad (8)$$

The circumferential stress readily follows from (4) and is given by

$$\sigma_\theta = (1-n_1)Ar^{-n_1} + (1-n_2)Br^{-n_2}. \quad (9)$$

The displacement can be conveniently deduced from the circumferential strain ε_θ as $u = r\varepsilon_\theta$. By substituting (7) and (9) into the second of (2), the circumferential strain is found to be

$$\varepsilon_\theta = \frac{1}{2\mu^b}\left(\frac{b}{r}\right)^m\left(\eta_1 Ar^{-n_1} + \eta_2 Br^{-n_2}\right), \quad (10)$$

with the parameters

$$\eta_1 = 1-2\upsilon-(1-\nu)n_1,$$

$$\eta_2 = 1-2\upsilon-(1-\nu)n_2. \quad (11)$$

Therefore, the radial displacement is

$$u = \frac{b^m}{2\mu^b}\left(\eta_1 Ar^{-m_1} + \eta_2 Br^{-m_2}\right), \quad (12)$$

where

$$m_1 = m+n_1-1, \; m_2 = m+n_2-1. \quad (13)$$

For a uniform material ($m = 0$), the parameters are $n_1 = 0$, $n_2 = 2$, $m_1 = -1$, and $m_2 = 1$.

4. BOUNDARY CONDITIONS

If the uniform pressures p and q are applied at the inner and outer boundary, the boundary conditions are $\sigma_r(a) = -p$ and $\sigma_r(b) = -q$. The integration constants in (7) are then

$$A = \frac{pc^{n_2}-q}{1-c^s}b^{n_1}, \; B = \frac{pc^{n_1}-q}{1-c^{-s}}b^{n_2}, \quad (14)$$

so that the radial and hoop stresses can be expressed as

$$\sigma_r(r) = \frac{pc^{n_2}-q}{1-c^s}\left(\frac{b}{r}\right)^{n_1} + \frac{pc^{n_1}-q}{1-c^{-s}}\left(\frac{b}{r}\right)^{n_2}, \quad (15)$$

$$\sigma_\theta(r) = (1-n_1)\frac{pc^{n_2}-q}{1-c^s}\left(\frac{b}{r}\right)^{n_1} + (1-n_2)\frac{pc^{n_1}-q}{1-c^{-s}}\left(\frac{b}{r}\right)^{n_2}. \quad (16)$$

The corresponding radial displacement is

$$u(r) = \frac{b}{2\mu^b}\left[\eta_1\frac{pc^{n_2}-q}{1-c^s}\left(\frac{b}{r}\right)^{m_1} + \eta_2\frac{pc^{n_1}-q}{1-c^{-s}}\left(\frac{b}{r}\right)^{m_2}\right]. \quad (17)$$

5. INCOMPRESSIBLE FUNCTIONALLY-GRADED CYLINDER

If the considered functionally-graded material is also elastically incompressible ($\upsilon = 1/2$), then $s = |2 — m|$. For example, if $m < 2$ there follows: $n_1 = 0$, $n_2 = 2 — m$, $m_1 = m — 1$, $m_2 = 1$, $n_1 = 0$, and $n_2 = (m — 2)/2$. Consequently, in this case,

$$\sigma_r(r)=\frac{pc^{2-m}-q}{1-c^{2-m}}+\frac{p-q}{1-c^{m-2}}\left(\frac{b}{r}\right)^{2-m}, \quad (18)$$

$$\sigma_\theta(r)=\frac{pc^{2-m}-q}{1-c^{2-m}}+(m-1)\frac{p-q}{1-c^{m-2}}\left(\frac{b}{r}\right)^{2-m}, \quad (19)$$

and

$$u(r)=\frac{b}{4\mu^b}(m-2)\frac{p-q}{1-c^{m-2}}\frac{b}{r}. \quad (20)$$

If $m = 1$, the hoop stress σ_θ is uniform throughout the cylinder.

5.1. Incompressible material with $m = 2$

For this particular type of the material ($\upsilon = 1/2$ and $m = 2$), the governing differential equation for the radial stress is, from (6),

$$r^2\frac{d^2\sigma_r}{dr^2}+r\frac{d\sigma_r}{dr}=0, \quad (21)$$

which has the solution

$$\sigma_r(r)=\frac{q-p}{\text{In}\, c}\text{In}\left(\frac{r}{b}\right)-q. \quad (22)$$

The corresponding hoop stress is

$$\sigma_\theta(r)=\frac{q-p}{\text{In}\, c}\text{In}\left(\frac{er}{b}\right)-q, \quad (23)$$

where e is the Naperian base of the natural logarithm.

The magnitude of the maximum shear stress is constant throughout the cylinder and equals to

$$\tau_{\max}=\frac{|\sigma_r-\sigma_\theta|}{2}=-\frac{|p-q|}{2\text{In}\, c}. \quad (24)$$

The hoop strain is

$$\varepsilon_\theta(r)=\frac{q-p}{4\mu^b\text{In}\, c}\left(\frac{b}{r}\right)^2, \quad (25)$$

so that the displacement varies with the radial distance according to

$$u(r)=r\varepsilon_\theta(r)=\frac{q-p}{4\mu^b\text{In}\, c}\frac{b^2}{r}. \quad (26)$$

In particular, if equal pressures are applied at both boundaries ($p = q$), no displacement takes place in the cylinder made of this particular functionally-graded incompressible material. For a thin-walled cylinder, with the thickness $\delta = b — a \ll b$, the approximation In $c \approx -\delta/b$ can be used. For a very thick cylinder ($b \gg a$), In $c \to\infty$ which gives rise to stress amplification.

6. CONCLUSION

We have presented an analysis of elastic response of a pressurized hollow cylinder whose stiffness is nonuniform in the radial direction according to the power law relationship $\mu(r) = \mu^b(r/b)^m$, where b is the outer radius of the cylinder. The stress components are expressed as linear combinations of two power functions of the radial coordinate r, whose exponents n_1 and n_2 depend on the Poisson ratio υ and the nonuniformity parameter m. The corresponding exponents in the displacement expression m_1 and m_2 also depend on υ and m only. Some distinct features of mechanical response are noted for incompressible functionally-graded material with the quadratic dependence of the elastic stiffness on the radial distance. It is shown that the maximum shear stress is constant throughout the cylinder, while the displacement decreases as b^2/r. No displacement takes place under equal pressures applied at both boundaries. The obtained results may be of interest for the optimization of material properties of functionally-graded materials.

7. ACKNOWLEDGMENTS

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О МЕХАНИЧКОМ ОДГОВОРУ ФУНКЦИОНАЛНО ГРАДИЈЕНТНОГ ЦИЛИНДРА ПОД ПРИТИСКОМ

Сажетак: Разматра се функционално градијентни цилиндар под притиском чији модули еластичности варирају са радијалном удаљености по степеној функцији. Уочене су неке специфичне карактеристике механичког понашања некомпресибилног функционално градијентног материјала чији је степен радијалне нехомогености једнак два. Показано је да је максимални напон смицања константан у цијелом цилиндру, док је помјерање пропорционално са 1/r уздуж радијалног правца. Помјерање је једнако нули у цијелом цилиндру када се на његову унутрашњу и спољашњу површину примијени једнаки притисак.

Кључне ријечи: еластичност, функционално градијентни материјал, нестишљивост, цилиндар под притиском, радијална нехомогеност, амплификација напона.