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INFLUENCE OF ANISOTROPY AND MAGNETIC FIELD ON THERMAL ENTANGLEMENT IN HEISENBERG MODEL AND THERMODYNAMIC ANALYSIS OF MODEL

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Abstract: The thermal entanglement in a two-qubit anisotropic Heisenberg XXZ system, also XYZ system, with Dzyaloshinskii-Moriya (DM) couplings in an inhomogeneous magnetic field, was studied. The effects of these two kinds of anisotropies on the thermal entanglement have been studied in detail in the concept for concurrence, the measure of entanglement. The analytical expressions of concurrence are obtained for this model. It is found that the DM interaction can enhance thermal entanglement and can be efficiently controlled by the DM interaction parameter and ehchange interaction J_x, J_y and J_z. When D is large enough, the entanglement can exist for larger temperatures and strong magnetic field. We also analysed thermodynamic properties of Heisenberg model and the most important results were shown in the paper.

Keywords: Thermal entanglement, Heisenberg XXZ and XYZ model, DM interaction.

1. INTRODUCTION

One of the most intriguing phenomena, whose existence is predicted by quantum mechanics, is entanglement. This concept was introduced by E. Schrodinger [1], for unusual quantum correlations that appears in a well-known thought experiment of Einstein, Podolsky and Rosen (EPR experiment) [2]. In this experiment, these three physicists showed (based on the quantum mechanics formalism) the existence of non-local quantum object, consisting of two or more parts. The path to the solution of the formed paradox (EPR paradox) was given by Bell [3]. He suggested his own statistical inequalities, and they do not satisfy non-local theories such as quantum mechanics. Experiments designed to test the entanglement [4−6] confirmed the correctness of the quantum mechanics. In this way, concept of the entanglement has become a physical reality that can not be modeled by any classical theory.

Entangled states have become a basic component in phenomena such as quantum teleportation [7], quantum cryptography [8], etc. Nowadays, because of the fact that theory of the entangled states is based on most fundamental ideas of quantum mechanics, exploring the idea of entangled states affect almost all areas of modern physics such as: atomic physics, quantum optics, solid state physics, nuclear and electromagnetic resonance spectroscopy, superconductor physics, etc.

Today, it turns out that spin systems at very low temperatures are very promising systems for the development of quantum information theory [10]. Important results were obtained for one dimensional anisotropic models [9], where the entanglement occurs between neighbouring spins. Formalism of thermal entanglement was implemented in Heisenberg's anisotropic model, with spin $S=\frac{1}{2}$, frustrated by DM interaction. In this paper, the concurrence C, as the quantitative measure of the entanglement of the system, was defined [10,14]. The value of concurrence is between zero and one, where, when it is equal to zero, the system is not in an entangled phase, and, when it is equal to the one, the system is maximally entangled. When $0<C<1$, the system is in an entangled phase to some extent. For mixed states, conccurrence of the state is given by $C(\rho)=\max\{2\lambda_{max}-\sum_{i=1}^{4}\lambda_i\,,0\}$, where λ_i's are square roots of eigenvalues of the matrix $R=\rho S\rho^* S$, ρ is density matrix, $S=\sigma_1{}^y\otimes\sigma_1{}^y$ and $\sigma_j{}^y$ (j=1,2) represent the Pauli matrices. It is very

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important to see how the system behaves in the external magnetic field [11], because the magnetic field is addressing the qubits of the system. A thermodynamic analysis of the model was performed too, in order to detect and compare the critical temperature of the entanglement, and Neel temperature in antiferromagnetic chain.

2. HEISENBERG XYZ TWO-QUBIT MODEL IN THE PRESENCE OF AN INHOMOGENEOUS MAGNETIC FIELD

In condensed matter systems, like Heisenberg spin chains, it is always possible to achieve an inhomogeneous Zeeman coupling [10,11,13,15]. Heterostructures of the condensed matter are usually inhomogeneous, in the presence of magnetic impurities such as defects. One of the main challenges is to realize the identical qubits in these systems [8,12]. The construction of approximately same systems in semiconductor technology has always been difficult - it still is, but they are extremely important in quantum technology [16].

In this part of paper, we consider two-qubit system in an inhomogeneous magnetic field [11,15], as well as the characteristics of the entanglement of these systems. At extremely low temperature, we assume that this qubit system is in the ground state [11]. Therefore, we are studying properties of ground state of entanglement. Of course, the real physical system is always on the non-zero temperature and consists of a mixture of entangled and unentangled states [11], which are dependent on temperature [11,15,16]. Since this is a two-qubit system, consisting of two spins, the results that will be shown are more relevant to the problem of quantum computer construction, than the interpretation of the problem of quantum phase transitions which requires treating an infinitely long spin chain.

Among the many concepts that are considered for the realization of two-qubit systems, the approach based on semiconductor quantum dots offers a great advantage in making miniature versions of quantum computers. Quantum dots [22] are very small particles in the semiconductor compound. They were discovered in condensed matter systems in 1980 by the Russian physicist Alexei Ekimov [20,21]. These are the crystals of only few nanometers and represent the central topic of nanotechnology (Figure 1) [21].

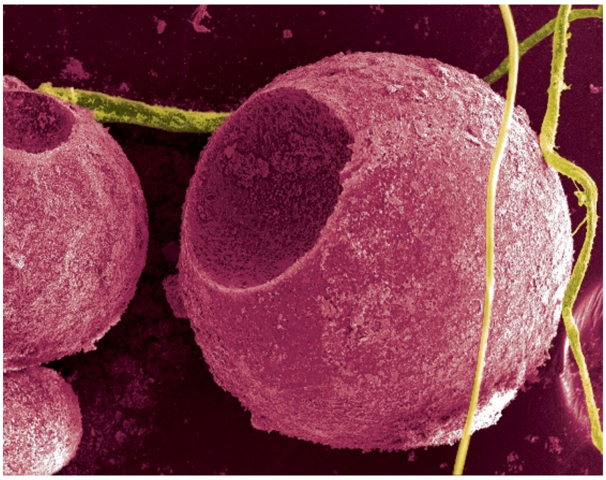

Figure 1. Quantum dots in semiconductor CdS crystal [21]

They have ability to emit radiation of a specific frequency if the electrical current or light by external sources is applied to them [16,20]. By applying low voltages to electrodes, it is possible to create a current that flows through quantum dots and precise measurements in spin systems can be made. With several entangled quantum points (etc. two-qubit system), we can get a flexible way of doing quantum operations, calculations and, in this way, we are able to construct a flexible quantum computer [17,18].

We consider the two-qubit system, consisting of two electrons, limited by two quantum dots (QD's) [11]. If the qubit is represented by one electron in a quantum dot, it can be manipulated by external devices [11]. Heisenberg model (the simplest model for the exploring of spin chain properties) is convenient for modeling two-qubit systems that involve spin-orbital interaction (SO interaction). SO interaction in nanostructures is studied using the quantum optics method [22]. This interaction creates a new type of anisotropy and is called spin - orbital anisotropy [11,15, 23]. The effect of SO interactions on the two-qubit XX system in the absence of a magnetic field was studied by a great number of physicists [11,15]. It has been shown that the critical temperature of entanglement increases with an increase in the absolute value of a parameter that characterizes the SO interaction. Also, it turns out that the thermal entanglement in this defined system is the same for ferromagnetic and antiferromagnetic order [11,15, 23].

In this paper, SO interaction involving second-order coupling is not taken into account. Only first - order was considered, which is proportional to coupling through exchange interaction. One of the first - order SO interactions is DM interaction [15].

3. THE MODEL AND THE HAMILTONIAN

The Hamiltonian of a two-qubit anisotropic Heisenberg XYZ model in the presence of an inhomogeneous magnetic field and spin-orbit interaction is defined as [11]

$$H = \frac{1}{2}\left(J_x \sigma_1{}^x \sigma_2{}^x + J_y \sigma_1{}^y \sigma_2{}^y + J_z \sigma_1{}^z \sigma_2{}^z\right) + \boldsymbol{B_1} \cdot \boldsymbol{\sigma_1} + \boldsymbol{B_2} \cdot \boldsymbol{\sigma_2} + \boldsymbol{D} \cdot (\boldsymbol{\sigma_1} \times \boldsymbol{\sigma_2}) + \delta \boldsymbol{\sigma_1} \cdot \overline{\boldsymbol{\Gamma}} \cdot \boldsymbol{\sigma_2}$$

where $\boldsymbol{\sigma_j} = (\sigma_j{}^x + \sigma_j{}^y + \sigma_j{}^z)$ is the vector of Pauli matrices, $\boldsymbol{B_j}$ (*j=1,2*) is the magnetic field acting on the site *j*, J_i (*i=x,y,z*) are the real coupling coefficients, $\boldsymbol{D}$ is the Dzyaloshinski-Moriya interaction vector, which is first order in SO coupling, and $\overline{\boldsymbol{\Gamma}}$ is a symmetric tensor which is second order in SO coupling. For simplicity, we assume $\boldsymbol{B_j} = B_j \boldsymbol{z}$ such that $\boldsymbol{B_1} = B + b$ and $\boldsymbol{B_2} = B - b$, where *b* indicates the amount of inhomogeneity of the magnetic field. The vector $\boldsymbol{D}$ and the parameter δ are dimensionless. If we ignore the second – order SO coupling, then the above Hamiltonian can be expressed as

$$H = J\gamma(\sigma_1{}^+ \sigma_2{}^+ + \sigma_1{}^- \sigma_2{}^-) + (J + \mathrm{i}J_z D)\sigma_1{}^+ \sigma_2{}^- + (J - \mathrm{i}J_z D)\sigma_1{}^- \sigma_2{}^+ + \frac{J_z}{2}\sigma_1{}^z \sigma_2{}^z + \frac{B+b}{2}\sigma_1{}^z + \frac{B-b}{2}\sigma_2{}^z$$

where $J = \frac{J_x + J_y}{2}$ is the mean coupling coefficient in the XY plane, $\gamma = \frac{J_x - J_y}{J_x + J_y}$ specifies the amount of anisotropy in the XY plane,and $\sigma^{\pm} = \frac{1}{2}\left(\sigma_x \pm \mathrm{i}\sigma_y\right)$ are lowering and raising operators. This Hamiltonian, in standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ has the following matrix form:

$$H = \begin{pmatrix} \frac{J_z}{2} + B & 0 & 0 & J\gamma \\ 0 & -\frac{J_z}{2} + b & J + \mathrm{i}J_z D & 0 \\ 0 & J - \mathrm{i}J_z D & -\frac{J_z}{2} - b & 0 \\ J\gamma & 0 & 0 & \frac{J_z}{2} - B \end{pmatrix}$$

We obtain the eigenstates of this Hamiltonian

$$|\Psi_1\rangle = M^-\left(\frac{B-\eta}{J\gamma}|00\rangle + |11\rangle\right)$$
$$|\Psi_2\rangle = M^+\left(\frac{B+\eta}{J\gamma}|00\rangle + |11\rangle\right)$$
$$|\Psi_3\rangle = N^-\left(\frac{B-\xi}{J-\mathrm{i}JD_z}|01\rangle + |10\rangle\right)$$
$$|\Psi_4\rangle = N^+\left(\frac{B+\xi}{J-\mathrm{i}JD_z}|01\rangle + |10\rangle\right)$$

where

$$M^{\pm} = \frac{1}{\sqrt{1 + \left(\frac{B \pm \eta}{J\gamma}\right)^2}}$$
$$N^{\pm} = \frac{1}{\sqrt{1 + \frac{(b \pm \xi)^2}{J^2 + (J_z D)^2}}}$$
$$\eta = \sqrt{B^2 + (J\gamma)^2}$$
$$\xi = \sqrt{b^2 + J^2 + (J_z D)^2}$$

Also, we calculate the eigenvalues of the Hamiltonian

$$E_{1,2} = \frac{J_z}{2} \pm \eta$$
$$E_{1,2} = -\frac{J_z}{2} \pm \xi$$

By using the definition of partition function, we calculated the partition function of the system

$$Z = \mathrm{Tr}\left(e^{\beta H}\right) = 2e^{\frac{\beta J_z}{2}}\left(\cosh(\beta\xi) + e^{-\beta J_z}\cosh(\beta\eta)\right)$$

By using the spectral decomposition theorem [12], we obtained density matrix of the system ρ

$$\rho = \begin{pmatrix} \mu_+ & 0 & 0 & \nu \\ 0 & w_1 & z & 0 \\ 0 & z^* & w_2 & 0 \\ 0 & \nu & 0 & \mu_- \end{pmatrix}$$

We defined notation in density matrix as

$$\mu_{\pm} = \frac{e^{-\frac{\beta J_z}{2}}}{Z}\left(\cosh(\beta\eta) \mp \frac{B}{\eta}\sinh(\beta\eta)\right)$$

$$w_{1,2} = \frac{e^{\frac{\beta J_z}{2}}}{Z}\left(\cosh(\beta\xi) \mp \frac{b}{\eta}\sinh(\beta\xi)\right)$$

$$\nu = -\frac{J\gamma e^{-\frac{\beta J_z}{2}}}{Z\eta}\sinh(\beta\eta)$$

$$z = -\frac{(J + \mathrm{i}J_z D)e^{\frac{\beta J_z}{2}}}{Z\xi}\sinh(\beta\xi)$$

In order to calculate concurrence of the system [10], first we must find following matrix $R = \rho S \rho^* S$ where $S = \sigma^y \otimes \sigma^y$. Square roots of eigenvalues of matrix R are

$$\lambda_{1,2} = \left|\sqrt{w_1 w_2} \pm |z|\right|$$

$$\lambda_{3,4} = \left|\sqrt{\mu_1\mu_2} \mp \nu\right|$$

Without loss of generality, we assume that $J>0$ and $\gamma>0$, as long as concurrence is invariant to sign change $J \to -J$ and $\gamma \to -\gamma$.

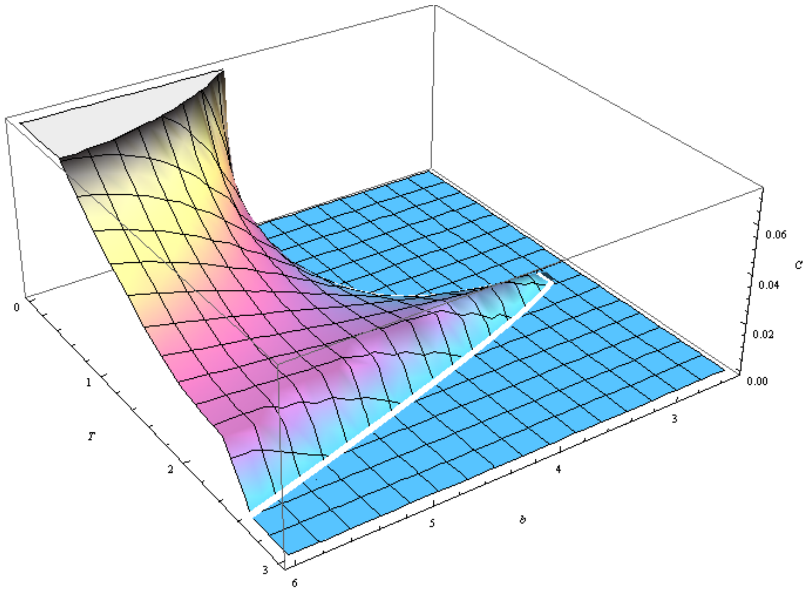

Figure 2. Concurrence of the system as a function of temperature T and inhomogenity parameter b for following values of parameters J=1, J_z=0.5, B=5 and γ=0.3

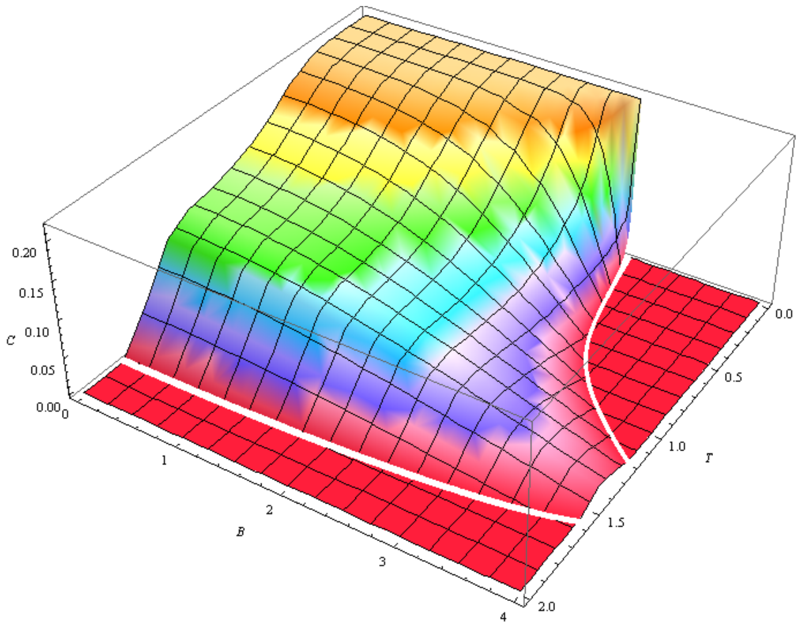

Figure 3. Concurrence of the system as a function of temperature T and magnetic field strength B for following values of parameters J=1, J_z=0.5, b=2, D=3 and γ=0.3

As we now have a magnetic field in the system, we will analyse the functional dependence of concurrence on the strength and parameter of the magnetic field inhomogenity.

From figures 2 and 3, we can conclude that the inhomogenity parameter and magnetic field strength can be good control parameters of the entanglement of the system. Of course, we confirmed the expectation that concurrence decreases with increasing temperature of the system. Also, concurrence of the system is decreasing function of magnetic field strength *B,* but it is increasing function of inhomogenity parameter *b*. These three parameters (*T,B,b*) can be controlled in experiment. The exchange interaction parameter can not be experimentally controlled. Changing the value of the exchange interaction in order to control the entanglement of the system is purely theoretical.

4. THERMODINAMIC ANALYSIS OF THE MODEL

In this part of the paper, we will discuss thermodynamic aspects of two-qubit anisoptopic Heisenberg model. Thermodynamic functions as: $\langle S_z \rangle$, internal energy, specific heat etc. will be calculated and ploted. We observe the same system as was discussed before. First, we calculate average S_z component of the spin:

$$\langle S_z \rangle = \mathrm{Tr}(\rho S_z) = (\mu_+ - \mu_-) - (w_1 - w_2)$$

Average value $\langle S_z \rangle$ will be equal to zero only if system is not in magnetic field, but when system is in an (in)homogenous magnetic field, $\langle S_z \rangle \neq 0$. Internal energy is defined as

$$U = \mathrm{Tr}(\rho\mathrm{H}) = \mu_+ \left(\frac{J_z}{2} + B\right) + 2\nu J\gamma$$
$$-\frac{J_z}{2}(w_1 + w_2) + b(w_1 - w_2) + J(z + z^*)$$
$$+\mathrm{i}J_z D(z^* - z) + \mu_- \left(\frac{J_z}{2} - B\right)$$

Specific heat is defined as

$$C_V = \frac{\partial U}{\partial T}$$

We can see in the figure 5 that we have two peaks. First of them comes from non-magnetic excitation, and we attribute them to existence of Kagoma clusters [19] (characteristic for antiferomagnetic systems). Presence of the DM interaction does not change the general structure significantly, but there is a slight change in temperature dependence of specific heat C_V . The second peak is important to us, because this peak corresponds to the phase transition from antiferomagnetic to paramagnetic phase. The Neel temperature is above $T_N \approx 2.5\ K$, and the critically entangled temperature is above $T_C \approx 1.7\ K$ (the condition for obtaining this temperature is that the concurrence is equal to zero). As we can see $\frac{T_N}{T_C} \approx 32\%$, and this means that the system will undergo a transition to a non-entangled phase and then another one to a paramagnetic phase.

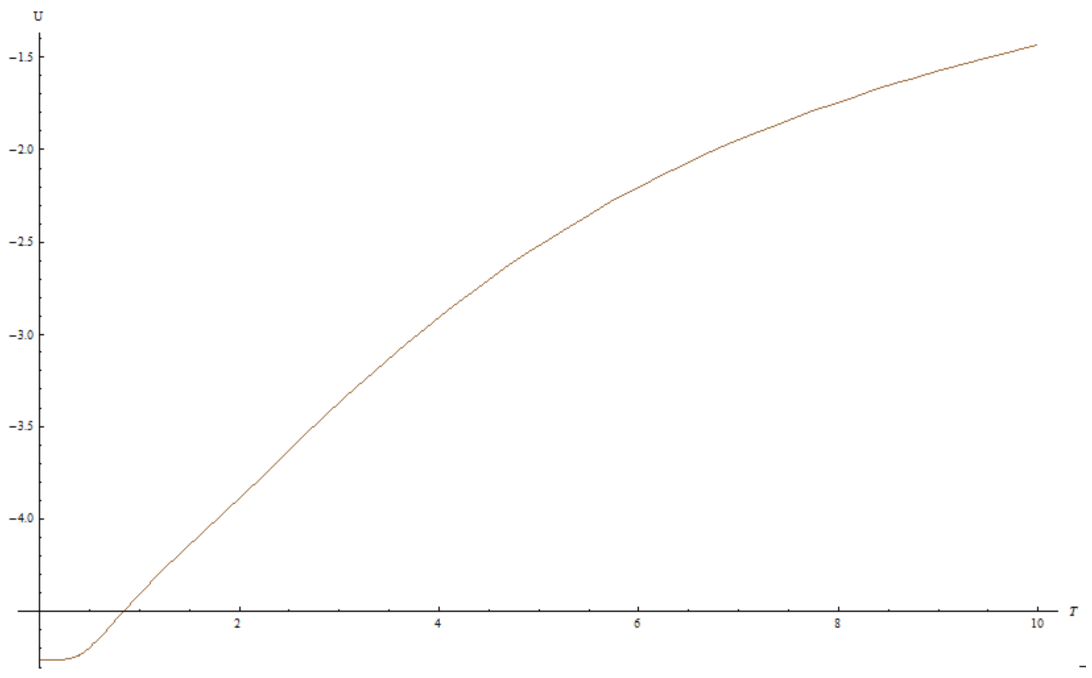

Figure 4. Internal energy as a function of temperature for parameters J=1, γ=0.3, J_z=0.5, B=5, b=2 and D_z=3

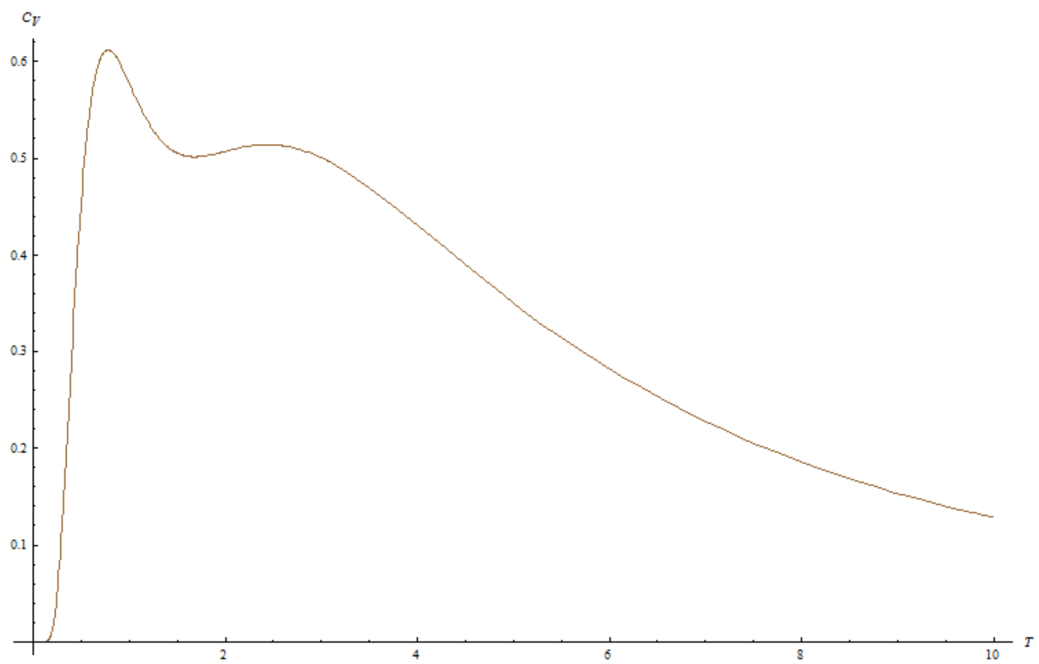

Figure 5. Specific heat as a function of temperature for parameters J=1, γ=0.3, J_z=0.5, B=5, b=2 and D_z=3

5. CONCLUSIONS

The thermal entanglement in a two-qubit anisotropic Heisenberg *XXZ* system, also *XYZ* system, with Dzyaloshinskii-Moriya (DM) couplings in an inhomogenous magnetic field, was studied. The effects of these two kinds of anisotropies on the thermal entanglement have been studied in detail in the concept for concurrence, the measure of entanglement. It is found that the DM interaction can enhance thermal entanglement and can be efficiently controlled by the DM interaction parameter and ehchange interaction J_x, J_y and J_z. When *D* is large enough, the entanglement can exist for larger temperatures and strong magnetic field. This means that the entanglement of the system can be controlled using four parameters (T,B,b,D). We also analysed thermodynamic properties of Heisenberg model. We observe that the presence of the DM interaction does not change the general structure significantly, but there is a slight change in temperature dependence of specific heat C_V. We obtained that when system, described by Heisenberg model, is in inhomogenous magnetic field, system will undergo a transition to a non-entangled phase and then another one to a paramagnetic phase.

6. ACKNOWLEDGMENT

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7. REFERENCES

[1] E . Schrödinger, Naturwissenschaften 23, 807, 823, 844 (1935); (Schrödinger E, Proc. Am. Phil. Soc. 124, 323 (1980) Translated into English).

[2] A. Einstein, B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev., Vol. 47 (1935) 777.

[3] J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics, Vol. 1 (1964) 195.

[4] A. Aspect, P. Grangier, G. Roger, *Experimental Tests of Realistic Local Theories via Bell's Theorem*, Phys. Rev. Lett., Vol. 47, (1981) 460.

[5] A. Aspect, P. Grangier, G. Roger, *Experimental Test of Bell's Inequalities Using Time—Varying Analyzers*, Phys. Rev. Lett., Vol. 49, (1982) 1804.

[6] P. Kwiat, K. Mattle, H. Weinfurter et. al., *New High-Intensity Source of Polarization-Entangled Photon Pairs*, Phys. Rev. Lett., Vol. 75, (1995) 4337.

[7] C. H. Bennett, G. Brassard, C. Crepeau et. al., *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett., Vol. 70, (1993) 1895.

[8] C. Macchiavello, G. M. Palma, Anton Zeilinger, *Quantum Computation and Quantum Information Theory*, World Scientific 2001.

[9] L. Amico, R. Fazio, A. Osterloh et.al., *Entanglement in many-body systems*, Rev. Mod. Phys., Vol. 80 (2008) 517−576.

[10] D. C. Li, X. P. Wang, Z. L. Cao, *Thermal entanglement in the anisotropic Heisenberg XXZ model with the Dzyaloshinskii – Moriya interaction*, J. Phys.: Condens. Matter, Vol. 20 (2008) 1−5.

[11] F. Kheirandish, S. J. Akhtarshenas, H. Mohammadi, *Effect of spin - orbit interaction on entanglement of two - qubit Heisenberg XYZ systems in an inhomogeneous magnetic field*, Physical Review A, Vol. 77 (2008) 042309-1 - 042309-9.

[12] V. Vedral, *Introduction to Quantum Information Science*, Oxford University Press, (2006).

[13] L. Da-Chuang, C. Zhuo-Liang, *Thermal Entanglement in the Anisotropic Heisenberg XYZ Model with Different Dzyaloshinskii – Moriya Couplings*, CHIN. PHYS. LETT. Vol. 26 (2009) 020309-1.

[14] K. Berrada, A. Chafik, H. Eleuch, Y. Hassouni, *Concurrence in the framework of coherent states*, Quantum Information Processing 2010 Vol. 9 (2009) 14-24.

[15] G. H. Yang, W. B. Gao, L. Zhou, H. S. Song, *The Entanglement in Anisotropic Heisenberg XYZ Chain with inhomogeneous magnetic field*, *arXiv:quant-ph/0602051*

[16] D. Lossa, D. P. DiVincenzo, *Quantum Computation with Quantum Dots*, *arXiv:cond-mat/9701055*

[17] R. C. Ashoori, *Electrons in artificial atoms*, Nature, Vol. 379−6564 (1996) 413−419.

[18] M. A. Kastner, *Artificial Atoms*, Physics Today 46−1 (1993) 24–31.

[19] A. V. Syromyatnikov, S. V. Maleyev, *Double peak specific heat feature in frustrated antiferromagnetic clusters*, *arXiv:cond-mat/0304572*

[20] Nanosys Quantum Dot Pioneers, Quantum Dots, *http://www.nanosysinc.com/what-we-do/quantum-dots*, (2015).

[21] https://en.wikipedia.org/wiki/Quantum_dot

[22] A. Predojević, *Photon generation efficiency and entanglement in quantum dot systems*, Contemporary materials, Vol. VII−2 (2016) 162−165.

[23] M. Pantić, M. Rutonjski, P. Mali, M. P. Hrvojević, S. Radošević, *Anisotropy influence on the striped antiferromagnetism*, Contemporary materials, Vol. VII−2 (2016) 121−126.

УТИЦАЈ АНИЗОТРОПИЈЕ И МАГНЕТНОГ ПОЉА НА ТЕРМАЛНИ ЕНТАНГЛЕМЕНТ У ХАЈЗЕНБЕРГОВОМ МОДЕЛУ И ТЕРМОДИНАМИЧКА АНАЛИЗА МОДЕЛА

Сажетак: У овом раду је проучаван термални ентанглемент у двокубитном анизотропном Хајзенберговом XXZ и XYZ систему, осујећеног Ђалошински–Морија (ДМ) интеракцијом у нехомогеном магнетном пољу. Утицај ове две врсте анизотропије и термални ентанглемент анализиран је помоћу физичке величине конкурентност, као квалитативне мере ентанглемента. Испоставља се да ДМ интеракција може појачати својство ентанглемента, као изменска интеракција J_x, J_y і J_z. Када је ДМ интеракција довољно велика, као и јачина магнетног поља, ентанглемент може опстати на вишим температурама. Такође, у раду су анализирана термодинамичка својства Хајзенберговог модела, те су изложени најбитнији резултати рада.

Кључне речи: термални ентанглемент, Хајзенбергов XXZ и XYZ модел, ДМ интеракција.