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# NON-LYAPUNOV STABILITY OF SINGULAR SYSTEMS: CLASSICAL AND MODERN APPROACHES WITH APPLICATION TO AUTOMATIC DRUG DELIVERY

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**Summary:** In this paper sufficient conditions for both practical and finite time stability of linear singular continuous time delay systems were introduced. The singular and singular time delay systems can be mathematically described as $E\dot{\mathbf{x}}(t)=A\mathbf{x}(t)$ and $E\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)$, respectively. Analyzing finite time stability, the new delay independent and delay dependent conditions were derived using the approaches based on Lyapunov-like functions and their properties on the subspace of consistent initial conditions. These functions do not need to be positive on the whole state space and to have negative derivatives along the system trajectories. When the practical stability was analyzed, the approach was combined with classical Lyapunov technique to guarantee the attractivity property of the system behavior. Furthermore, an LMI approach was applied to obtain less conservative stability conditions. The proposed methodology was applied and tested on a medical robotic system. The system was designed for different insertion tasks playing important roles in automatic drug delivery, biopsy or radioactive seeds delivery. In this paper we have summarized different techniques for adequate modeling, control and stability analysis of the medical robots. The model of the robotic system, with the tasks described above, the entire system can be decomposed to the robotic subsystem and the environment subsystem. Modeling of the system by the method mentioned has been proved to be suitable when the force appears as a result of the interaction of the two subsystems. The mathematical model of the system has a singular characteristic. The singular system theory could be applied to the case described. It is well known that all mechanical systems have some delay. In that case a theory of singular systems with delayed states may be applied, as well. For the second phase in which there is no interaction, the dynamic behavior can be analyzed by the classic theory.

**Keywords:** time delay systems, singular systems, robotics in medicine.

## 1. INTRODUCTION

It was noticed that the characteristics of the dynamic and static state should be considered at the same time for some systems. Singular systems (also referred to as degenerate, descriptor, generalized, differential-algebraic or semi-state systems) are systems whose dynamic is governed by the complexity of algebraic and differential equations. Recently, many researchers have paid much attention to singular systems and they have accomplished numerous valuable conclusions.

The complex nature of singular systems generates many difficulties in the analytical and numerical solution of such systems, particularly during the control tasks.

Recently, the singular systems have been one of the major research fields of control theory.

During the past three decades singular systems have attracted significant attention due to

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the comprehensive applications in economics, as the Leontief dynamic model, in electrical applications using the theory described in [1], in mechanical models as in [2], *etc*. Singular systems in control theory have been initially discussed in [3] and [4]. The investigation of time delay systems has been carried out over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

It has been observed that a variety of singular systems is characterized by the phenomena of time delay. Such systems are called singular differential systems with time delay. These systems have many special characteristics. In order to mathematically describe those systems in more accurate manner, and to control them more effectively, this specific class of the singular systems was investigated in details. In this article, a new approach to the stability of the singular time delay systems is presented.

## 2. SYSTEM MODELING

In this section a procedure for the system modeling is described. A mathematical model of the presented medical robotic system was used to validate the main results and stability investigation.

The mathematical equations of the system were analyzed further and new delay-independent and delay-dependent conditions were implemented in practical stability analysis.

### 2.1. System description

The surgery module consists of 2 degree-of-freedom (DOF) ultrasound probe driver and 5DOF needling module, Figure 1.

The ultrasound (US) module can be translated and rotate independently by two DC servomotors fitted with high-resolution optical encoders and gearboxes.

In this study, we analyzed 5DOF needling module which consists of a gantry and needle driver.

Gantry connects the needle-driving module to the positioning platform.

The gantry has two translation motions and one rotational motion (pitching). Needle driver subsystem consists of a hollow needle (cannula) and solid needle (stylet) driven separately by two DC servomotors.

The cannula rotates continuously or partially using another tiny DC motor. The main task of these parts is to deliver the exactly prescribed dose of radioactive seeds into human prostate with high precision level.

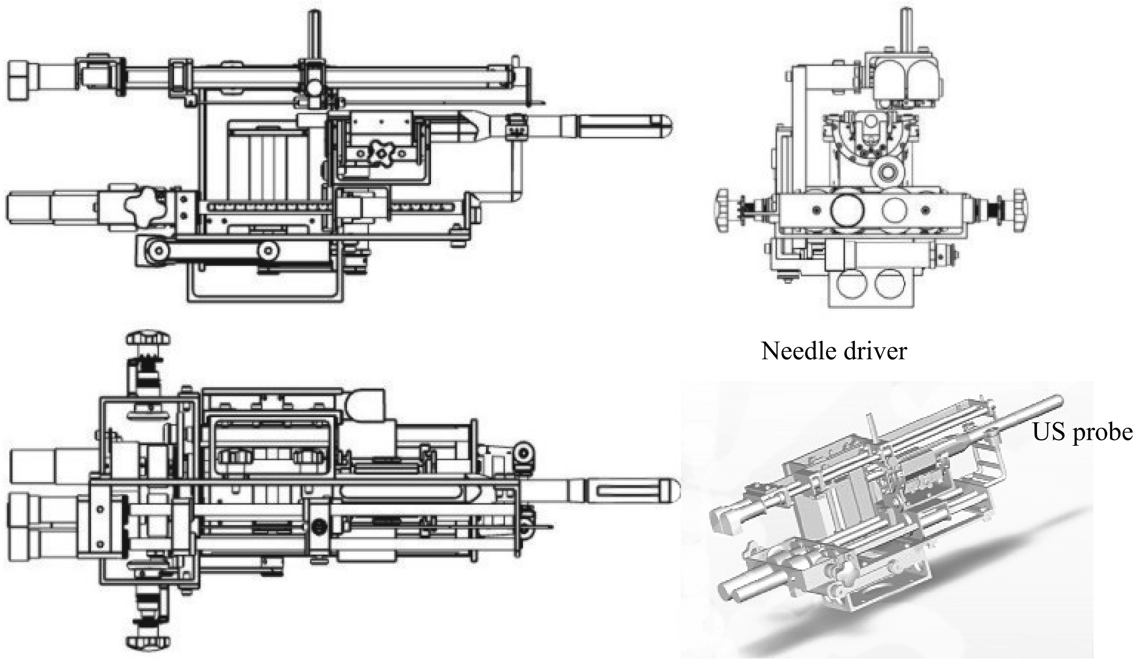


*Figure 1. Video-guided robotic system for insertion tasks. The proposed methodology was tested on this system. Surgery module: consists of 2DOF ultrasound probe driver and 5DOF needling module. Needling module consists of gantry and needle driver. Mathematical model of the system was singular system as in equations* (3-4)

Seeds are delivered through cannula.

During the operation, stylet is pushing the seeds through cannula according to control algorithm and the prescribed surgery plan. Also, the system is designed to take ultrasound images during the operation, to update the real-time radiation dose distribution, seed position and number of needles to be inserted into prostate, depending of surgery plan.

Dedicated software for 3D imaging and control is developed to support surgery procedure, [5-6].

### 2.2. Mathematical modeling

The following notation has been used:

$\Re$ Real vector space
$\mathbb{C}$ Complex vector space
$I$ Identity matrix
$F = \left(f_{ij}\right) \in \Re^{n \times n}$ Real matrix
$F^T$ Transpose of matrix $F$
$F > 0$ Positive definite matrix
$F \geq 0$ Positive semi definite matrix
$\lambda(F)$ Eigenvalue of matrix $F$
$\|F\| = \sqrt{\lambda_{max} \max\left(A^T A\right)}$ Euclidean matrix norm of $F$

As suggested [7], the most accurate mathematical model for the medical robots should include dynamics of the system due to interaction between the robot and the surface.

General guidelines for mathematical modeling together with the basic equations are presented.

The model of the manipulator with its constraints is shown in Figure 2.

Generally speaking, open kinematic chain with n joints is analyzed.

The generalized coordinates vector, denoted by $\mathbf{q}(t)$, has property $\mathbf{q}(t) \in \Re^n$, the contact force vector is denoted by $\mathbf{f}(t)$.

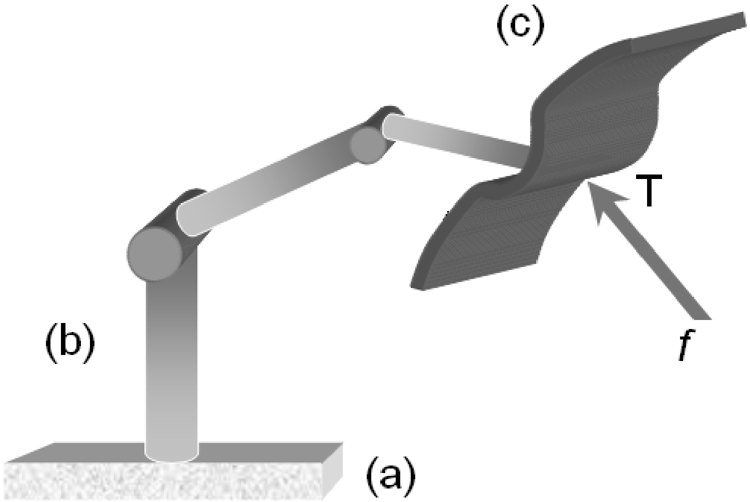


*Figure 2. Model of the constrained robotic system: a) fixed base, b) manipulator c) contact surface, T – contact point, f – contact force*

Force $\mathbf{f}(t) \in \Re^n$ appears when end-effector touches constraint surface *c*.

The differential equation which describes the influence on the contact force to the system is

$$M\left(\mathbf{q}(t)\right)\ddot{\mathbf{q}}(t) + \mathbf{g}\left(\mathbf{q}(t), \dot{\mathbf{q}}(t)\right) = \boldsymbol{\tau}(t) + J^T\left(\mathbf{q}(t)\right)\mathbf{f}(t). \tag{1}$$

$M\left(\mathbf{q}(t)\right) \in \Re^{n \times n}$ denotes inertia matrix function and $\mathbf{g}\left(\mathbf{q}(t), \dot{\mathbf{q}}(t)\right) \in \Re^n$ is vector function which describes Coriolis, centrifugal and gravitational effects. $\boldsymbol{\tau}(t)$ is torque vector of the joints, $\boldsymbol{\tau}(t) \in \Re^n$.

$J^T\left(\mathbf{q}(t)\right) \in \Re^{(n \times n)}$ is defined as Jacobian matrix function and $D(\cdot)$ is a gradient of constrained function.

The general dynamic equations for the robotic system in contact with environment is, as in [8]:

$$\begin{pmatrix} M\left(\mathbf{q}(t)\right) & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \ddot{\mathbf{q}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{g}\left(\mathbf{q}(t), \dot{\mathbf{q}}(t)\right) + J^T\left(\mathbf{q}(t)\right) D^T\left(\mathbf{h}(\mathbf{q})\right) \lambda(t) \\ \phi\left(\mathbf{h}(\mathbf{q})\right) \end{bmatrix} \tag{2}$$

Equation (2) consisted of $n$ differential equations and one algebraic equation with $(n+1)$ unknown value, $n$ generalized coordinates and scalar multiplier $\lambda(t)$.

$\phi(\mathbf{h}(\mathbf{q}))$ is an equation of contact surface, and $\mathbf{h}(\ )$ is a vector function.

Now it is possible to present the equation of the robotic system (1) which is in contact with the working environment in its state space form (3) with state vector $\mathbf{x}(t)$ and vector $\mathbf{d}(t)$ as a disturbance

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{d}(t), \tag{3}$$

where corresponding matrices have been defined as in [7].

Corresponding matrices are given by equation (4).

For the purpose of further analysis, we considered disturbance vector $\mathbf{d}=\mathbf{0}$.

$$E = \begin{pmatrix} I & 0 & 0 \\ 0 & M(\mathbf{q}_0) & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & I & 0 \\ \dfrac{\partial}{\partial \mathbf{q}}\left(\mathbf{g} - J^T D^T \lambda(t)\right)|_0 & 0 & J^T D^T\,|_0 \\ DJ\,|_0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix},\quad \mathbf{u}(t) = \delta\boldsymbol{\tau}(t),\quad \mathbf{d}(t) = \begin{bmatrix} 0 \\ \Delta\boldsymbol{\tau}(t) \\ 0 \end{bmatrix} \tag{4}$$

When time delay of moving system parts was taken into the account, the system (3) was represented as

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t) + \mathbf{d}(t). \tag{5}$$

System (5) represents the dynamics of the medical robot in Figure 1 with time delay in working regime. Further analysis was performed in free working regime, *i.e.* when all inputs have zero values.

## 3. STABILITY CONCEPTS

As far as practical problems are concerned, a matter of interest is not only the system stability (*e.g.* in the sense of Lyapunov), but also the bounds of system trajectories. A system could be stable but completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of the systems with respect to certain subsets of state-space, which are *a priori* defined for a given problem.

Besides that, it is of particular significance to consider the behavior of dynamical systems only over a finite time interval. These bound properties of system responses, *i.e.* solutions of system models, are important from the engineering point of view.

Realizing this fact, numerous definitions of the so-called technical and practical stability have been introduced in literature. Generally speaking, the definitions were essentially based on the predefined boundaries for the perturbation of initial conditions, and the allowable perturbation of the system response. In the engineering applications of control systems, this fact becomes important and sometimes crucial for the purpose of quantitative characterizing of the systems. In that case, the possible deviations of the system response need to be investigated in details. Thus, the analysis of these particular bound properties of the solutions presents an important step, which precedes the design of control signals, with finite time or practical stability taken into account. In this article time continuous systems have been considered.

The various notations of stability over a finite time interval for continuous time systems and constant set trajectory bounds were introduced in [9−11]. Another approach is based on a classical theory mostly used in deriving sufficient delay independent conditions of the finite time stability systems.

In the former case a new definition has been introduced based on the attractivity properties of the system solution which can be treated as analogous to the quasi-contractive stability as in [12−14].

In the following section, we have presented a novel approach to the stability of singular time delay systems.

The results have been directly expressed in terms of matrices $E$, $A_0$ and $A_1$ naturally occurring in the system model, equation (5).

In this approach there is no need to introduce any canonical form in the statement of the theorems.

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions [15]. This fact makes the construction of the Lyapunov and non- Lyapunov stability theory possible even for the *linear continuous singular time-delay systems* (LCSTDS).

Moreover, the attractive property is equivalent to the existence of symmetric positive definite solutions in a weak form of the Lyapunov matrix equation [15], incorporating conditions which refer to the solutions boundedness.

3.1. Preliminaries

The general expression of singular control systems with time delay can be written in its differential form as:

$$\begin{gathered} E(t)\dot{\mathbf{x}}(t)=\mathbf{f}\left(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\mathbf{u}(t)\right), \quad t\geq 0 \\ \mathbf{x}(t)=\boldsymbol{\varphi}(t),\ -\tau\leq t\leq 0 \end{gathered}, \tag{6}$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a state vector, $\mathbf{u}(t)\in\mathbb{R}^m$ is a control vector, $E(t)\in\mathbb{R}^{n\times n}$ is a singular matrix, $\boldsymbol{\varphi}\in\mathcal{C}=\left([-\tau,0],\ \mathbb{R}^n\right)$ is an admissible initial state functional, $\mathcal{C}=\left([-\tau,0],\ \mathbb{R}^n\right)$ is the *Banach* space of continuous functions mapping the interval $[-\tau,\ 0]$ into $\mathbb{R}^n$ with topology of uniform convergence.

The vector function satisfies:

$$\mathbf{f}(\cdot):\Im\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\rightarrow\mathbb{R}^n, \tag{7}$$

and it is assumed to be smooth enough to assure the existence and uniqueness of solutions over a time interval:

$$\Im=\left[t_0,\ (t_0+T)\right[\in\mathbb{R}, \tag{8}$$

as well as the continuous dependence of the solutions denoted by $\mathbf{x}(t,\ t_0,\mathbf{x}_0)$ with respect to $t$ and the initial data.
Quantity $T$ may be either a positive real number or the symbol $+\infty$, so that the finite time stability and practical stability can be treated simultaneously, respectively.

In general, it is not required that $\mathbf{f}(t,\ \mathbf{0},\ \mathbf{0})\equiv\mathbf{0}$, for an autonomous system, which means that the origin of the state space is not necessarily required to be an equilibrium state.

Let $\mathbb{R}^n$ denote the state space of a system given by (6) and $\|(\cdot)\|$ the Euclidean norm.

Let $V:\Im\times\mathbb{R}^n\rightarrow\mathbb{R}^n$, be the tentative aggregate function, so that $V(t,\mathbf{x}(t))$ is bounded and for which $\|\mathbf{x}(t)\|$ is also bounded.

Define the Eulerian derivative of $V(t,\mathbf{x}(t))$ along the trajectory of the system (6), with:

$$\dot{V}(t,\mathbf{x}(t))=\frac{\partial V(t,\mathbf{x}(t))}{\partial t}+\left[grad\ V(t,\mathbf{x}(t))\right]^T\Phi\,\mathbf{f}(\cdot). \tag{9}$$

where matrix $\Pi$, [16], is solution of the following matrix equation:

$$\left[grad\ V(t,\mathbf{x}(t))\right]^T=\left[grad\ V(t,\mathbf{x}(t))\right]^T\Phi E, \tag{10}$$

Obtaining this solution may be a tedious task since matrices in (10) are functional, [16].

For time-invariant sets it is assumed: $\mathcal{S}_{(\ )}$ is a bounded open set.

The closure and boundary of $\mathcal{S}_{(\ )}$ are denoted by $\bar{\mathcal{S}}_{(\ )}$ and $\partial\mathcal{S}_{(\ )}$, respectively, so: $\partial\mathcal{S}_{(\ )}=\bar{\mathcal{S}}_{(\ )}\setminus\mathcal{S}_{(\ )}$.

Let $\mathcal{S}_\beta$ be a given set of all allowable states of the system $\forall t\in\Im$.

Set $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha\subset\mathcal{S}_\beta$ denotes the set of all allowable initial states.

Sets $\mathcal{S}_\alpha$, $\mathcal{S}_\beta$ are connected and *a priori* known. $\lambda(\ )$ denotes the eigenvalues of matrix ( ).

$\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively.

For the further analysis we consider a linear continuous singular system with state delay, described by:

$$E\,\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau), \tag{11}$$

with a known compatible vector valued function of the initial conditions

$$\mathbf{x}(t)=\boldsymbol{\varphi}(t),\ -\tau\leq t\leq 0, \tag{12}$$

where $A_0$ and $A_1$ are the constant matrices of appropriate dimensions.

Moreover, we shall assume that $rank\ E=r<n$.

3.3. Basic definitions

Definition 1. Matrix pair $(E,\ A_0)$ is said to be regular if $\det(sE-A_0)$ is not identically zero, [17].

Definition 2. The matrix pair $(E,\ A_0)$ is said to be impulsive free if $degree\ \det(sE-A_0)=rank\,E$, [17].

The linear continuous singular time delay system (6) may have an impulsive solution. However, the regularity and the absence of impulses of the matrix pair $(E,\ A_0)$ ensure the existence and uniqueness of an impulse-free solution of the system.

The existence of the solutions is defined in the following *Lemma*.

Lemma 1. Suppose that the matrix pair $(E, A_0)$ is regular and impulsive free, then the solution to (11) exists and is impulse-free and unique on $[0, \infty[$, [17].

As a necessity for the system stability investigation there is a need to establish a proper stability definition.

Therefore, the following definition can be written.

Definition 3. (a) LCSTDS (11-12) is said to be regular and impulsive free, if the matrix pair $(E, A_0)$ is regular and impulsive free. (b) LCSTDS (11-12) is said to be stable, if for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\boldsymbol{\varphi}(t)$, $\sup_{-\tau \le t \le 0} \| \boldsymbol{\varphi}(t) \| \le \delta(\varepsilon)$ the solution $\mathbf{x}(t)$ of system (6) satisfies $\| \mathbf{x}(t) \| \le \varepsilon, \ \forall t \ge 0$.

Moreover, if $\lim_{t\to\infty} \| \mathbf{x}(t) \| \to 0$, the system is said to be asymptotically stable, [17].

## 4. MAIN RESULTS

Definition 4. Singular time delayed system (11-12) is a *finite time stable* with respect to $\{\alpha, \beta, \Im, R\}$, $\alpha < \beta$, and $R > 0$, if $\sup_{t\in[-\tau, 0]} \boldsymbol{\varphi}^T(t) E^T R E \boldsymbol{\varphi}(t) \le \alpha$ implies $\mathbf{x}^T(t) E^T R E \mathbf{x}(t) < \beta, \ \forall t \in \Im$.

Finally, by using matrix inequalities, we can derive the sufficient condition under which the system (11-12) will be regular, impulse free and finite time stable.

Theorem 1. Singular time delayed system (11−12) is *impulse free* and *finite time stable* with respect to $\left\{ \mathcal{S}_\alpha, \mathcal{S}_\beta, \Im, R, \| (\cdot) \|^2 \right\}$ $\alpha < \beta$ if, letting $PE = E^T R^{\frac{1}{2}} \Pi R^{\frac{1}{2}} E$, there exist a positive scalar $\wp > 0$ and two positive definite matrices $\Pi \in \mathbb{R}^{n\times n}, \Pi = \Pi^T > 0,$ , and $Q \in \mathbb{R}^{n\times n}$, such that the following conditions hold:

$$PE = E^T P^T > 0, \tag{13}$$

$$\Xi = \begin{pmatrix} \left( A_0^T P^T + PA_0 + Q \right) - \wp EP & PA_1 \\ A_1^T P^T & -Q \end{pmatrix} < 0, \tag{14}$$

and:

$$\left( \frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(\Pi)} + \tau \frac{\lambda_{\max}(Q)}{\lambda_{\min}(\Pi)} \right) e^{\wp t} < \frac{\beta}{\alpha}, \quad \forall t \in \Im . \tag{15}$$

Proof. Let us consider the following Lyapunov-like, aggregation function:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) PE\, \mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\vartheta , \tag{16}$$

Denoting by $\dot{V}(\mathbf{x}(t))$ time derivative of $V(\mathbf{x}(t))$ along the trajectory of system (11-12), it can be written:

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \dot{\mathbf{x}}^T(t) PE\, \mathbf{x}(t) + \mathbf{x}^T(t) PE\, \dot{\mathbf{x}}(t) + \frac{d}{dt} \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\vartheta \\
&= \mathbf{x}^T(t) \left( A_0^T P^T + PA_0 \right) \mathbf{x}(t) \\
&+ \mathbf{x}^T(t) PA_1 \mathbf{x}(t-\tau) + \mathbf{x}^T(t-\tau) A_1^T P^T \mathbf{x}(t) \\
&+ \mathbf{x}^T(t) Q \mathbf{x}(t) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \\
&= \boldsymbol{\zeta}^T(t) \Gamma \boldsymbol{\zeta}^T(t)
\end{aligned} \tag{17}$$

where:

$$\boldsymbol{\zeta}^T(t) = \left[ \mathbf{x}^T(t) \quad \mathbf{x}^T(t-\tau) \right], \qquad \Gamma = \begin{pmatrix} \left( A_0^T P^T + PA_0 + Q \right) & PA_1 \\ A_1^T P^T & -Q \end{pmatrix} . \tag{18}$$

From (13) and (16) it can be derived:

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \boldsymbol{\zeta}^T(t) \Gamma \boldsymbol{\zeta}(t) \\
&= \boldsymbol{\zeta}^T(t) \left( \Xi - \begin{pmatrix} -\wp PE & 0 \\ 0 & 0 \end{pmatrix} \right) \boldsymbol{\zeta}(t) \\
&= \boldsymbol{\zeta}^T(t) \Xi \boldsymbol{\zeta}(t) - \boldsymbol{\zeta}^T(t) \begin{pmatrix} -\wp PE & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\zeta}(t) \\
&= \boldsymbol{\zeta}^T(t) \Xi \boldsymbol{\zeta}(t) + \wp \mathbf{x}^T(t) PE\, \mathbf{x}(t) \\
&< \wp \mathbf{x}^T(t) PE\, \mathbf{x}(t) < \wp \mathbf{x}^T(t) PE\, \mathbf{x}(t) + \wp \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\vartheta \\
&= \wp \left( \mathbf{x}^T(t) PE\, \mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\vartheta \right) \\
&= \wp V(\mathbf{x}(t))
\end{aligned} \tag{19}$$

since $\boldsymbol{\zeta}^T(t) \Xi \boldsymbol{\zeta}(t) < 0$.

Multiplying (19) by $e^{-\wp t}$, it is obtained:

$$\frac{d}{dt} \left( e^{-\wp t} V(\mathbf{x}(t)) \right) < 0 . \tag{20}$$

Integrating (20) from $0$ to $t$, with $t \in \Im$, it follows:

$$V(\mathbf{x}(t)) < e^{\wp t} V(\mathbf{x}(0)) . \tag{21}$$

Consequently:

$$V\left(\mathbf{x}(0)\right)=\mathbf{x}^{T}(0)PE\,\mathbf{x}(0)+\int_{-\tau}^{0}\mathbf{x}^{T}(\vartheta)Q\,\mathbf{x}(\vartheta)\,d\vartheta \quad . \tag{22}$$

Since:

$$PE=E^{T}P^{T}=E^{T}R^{\frac{1}{2}}\,\Pi\,R^{\frac{1}{2}}E\,, \tag{23}$$

from (22) and first condition of *Definition* 4, it follows:

$$\begin{aligned}V\left(\mathbf{x}(0)\right)&=\mathbf{x}^{T}(0)E^{T}R^{\frac{1}{2}}\,\Pi\,R^{\frac{1}{2}}E\,\mathbf{x}(0)+\int_{-\tau}^{0}\mathbf{x}^{T}(\vartheta)Q\,\mathbf{x}(\vartheta)\,d\vartheta\\&\le\lambda_{\max}(\Pi)\mathbf{x}^{T}(0)E^{T}RE\,\mathbf{x}(0)+\lambda_{\max}(Q)\int_{-\tau}^{0}\boldsymbol{\varphi}^{T}(\vartheta)\boldsymbol{\varphi}(\vartheta)\,d\vartheta\\&\le\lambda_{\max}(\Pi)\cdot\alpha+\lambda_{\max}(Q)\cdot\alpha\int_{-\tau}^{0}d\vartheta=\alpha\left(\lambda_{\max}(\Pi)+\tau\cdot\lambda_{\max}(Q)\right)\end{aligned} \tag{24}$$

Furthermore, it can be calculated that:

$$\begin{aligned}V\left(\mathbf{x}(t)\right)&=\mathbf{x}^{T}(t)PE\,\mathbf{x}(t)+\int_{t-\tau}^{t}\mathbf{x}^{T}(\vartheta)Q\,\mathbf{x}(\vartheta)\,d\vartheta>\mathbf{x}^{T}(t)PE\,\mathbf{x}(t)\\&=\mathbf{x}^{T}(t)E^{T}R^{\frac{1}{2}}\,\Pi\,R^{\frac{1}{2}}E\mathbf{x}(t)>\lambda_{\min}(\Pi)\mathbf{x}^{T}(t)E^{T}RE\mathbf{x}(t)\end{aligned} \quad . \tag{25}$$

From (25) it is obvious:

$$\mathbf{x}^{T}(t)E^{T}RE\mathbf{x}(t)<\frac{1}{\lambda_{\min}(\Pi)}V\left(\mathbf{x}(t)\right)\,, \tag{26}$$

so combining (21), (24) and (26), leads to:

$$\mathbf{x}^{T}(t)E^{T}RE\mathbf{x}(t)<\frac{1}{\lambda_{\min}(\Pi)}e^{\wp t}V\left(\mathbf{x}(0)\right)<\alpha e^{\wp t}\,\frac{\lambda_{\max}(\Pi)+\tau\cdot\lambda_{\max}(Q)}{\lambda_{\min}(\Pi)}\quad . \tag{27}$$

Condition (15) and inequality (27) imply:

$$\mathbf{x}^{T}(t)E^{T}R\,E\,\mathbf{x}(t)<\beta,\quad\forall t\in\Im\,. \tag{28}$$

*Q.E.D.*

The finite time stability of the system with respect to *Definition* 4 was investigated.

For the numerical stability analysis, *Theorem* 1 was used.

The numerical values of matrices $A_{L0}$ and $A_{L1}$ are as follows:

## 5. DYNAMIC ANALYSIS

In this section, the dynamic analysis of the system (3) was performed.

$$A_{L0}=\begin{bmatrix}0&1&0&0&0&0&0&0&0&0\\0&-3.9e6&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&-3.9e6&8.42&-26.92&-1.5e\text{-}4&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&-3.27e3&2.8e\text{-}2&-4.6e4&-0.24&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&-4.1e\text{-}2&0&4.4e\text{-}3&-2.9e5&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&-4.2e6\end{bmatrix},\quad A_{L1}=\begin{bmatrix}0&1&0&0&0&0&0&0&0&0\\0&-2.33e5&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&-2.3e3&8.42&-4.28&-1.5e\text{-}3&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&-3.3e4&0.18e\text{-}1&-4.6e4&-0.32&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&-0.44e\text{-}1&0&2.4e\text{-}2&-2.3e4&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&-3.2e5\end{bmatrix}.$$

System matrices $A_{L0}$ and $A_{L1}$ were calculated for the system with feedback, as in Fig.3.

For this example it was adopted $\alpha$ = 0.8 and $\beta$= 3.42.

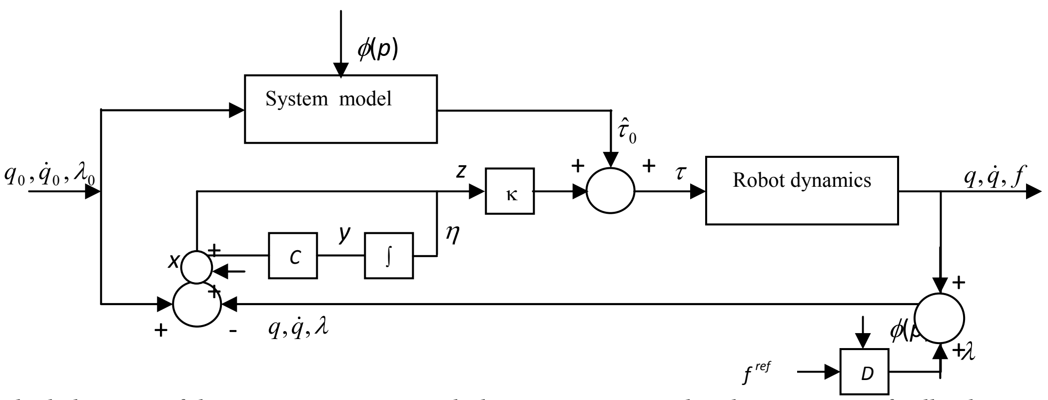


*Figure* 3. *Block diagram of the system in contact with the environment and with appropriate feedback control signals*

The singular matrix $E$ was calculated from (4) using described modeling procedure.

Using control low:

$$\mathbf{u}(t) = -K\,C\,\mathbf{x}(t), \tag{29}$$

where $K$ is system gain matrix and $C$ system output matrix.

$K$ is diagonal matrix and their elements are position and velocity gains, $K = diag\{K_p, K_v\}$.

Gain values for each segment can be calculated using actuators characteristics.

Detailed explanations for this procedure can be found in [18].

Using the control law (29), (4) and (30), it is possible to calculate eigenvalues of the system (5).

$$\det(A_L - KC - sE) = K\prod_{j=1}^{N}\left(s_j^0 - s\right) = \det\left(\left(A_{0L} + A_{1L}e^{-s\tau}\right) - CK - sE\right) = \det\left(\left(A_{0L} + \bar{A}_{1L}\right) - CK - sE\right) \tag{30}$$

where $A_L = \left(A_{0L}\ \bar{A}_{1L}\right)$.

It is to be noted that the eigenvalues are not constant values, but they depend on the specific value of time delay. Simulating system for $\tau$ = 200 ms, one sets possible eigenvalues that guarantee system stability is $\sigma(\lambda)$ = {8.3, 2.8*e*5, 7.2*e*5, 1.2, 6.4*e*5, 1245, 12,4, 2.4*e*5, 4234, 4.1*e*6 }.

Adopting the adequate matrices from the conditions of *Theorem* 1, using equations (13), it is possible to calculate scalar $\wp$ for which the system stability holds. It can be noticed that $\lambda_{\max}(\Pi)$ = 4.4*e*7. In this case, it was derived that the stability holds for $\wp > 0.5$, and system is finite time stable for $t \in\ ]\,0, 4s]$.

Figure 4. shows the representative system norms of both stabilized and non-stabilized systems, whereas Figure 5. represents the corresponding trajectories of the system related to the ones presented in Figure 4.

It can be noticed that the system shows its finite time stability up to 4 s.

It was observed that the non-stabilized system (open loop system) was finite time stable at the interval [0, 1.3 s] and at interval [1.3 4 s] with respect to $\beta$ = 3.42 (Figure 4., magenta curve).

Applying stabilization control law (29) resulted both in the asymptotic stability and the finite time stability on the interval [0.0 4.0 s], for all points as it was requested as a synthesis goal.

Finally, it was observed that when condition of *Definition* 4 and previously calculated conditions from *Theorem* 1 was not satisfied, the system showed instable behavior.

Figure 6 represents the case of the system trajectories for some $\beta$, as in condition (28).

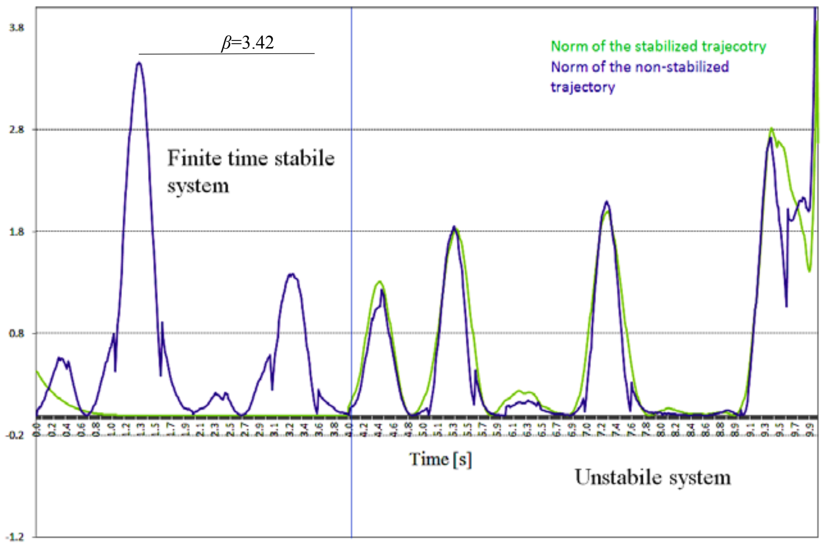


*Figure 4. The norms of the stabilized closed-loop system trajectories and non-stabilized system trajectories – a representative case -*

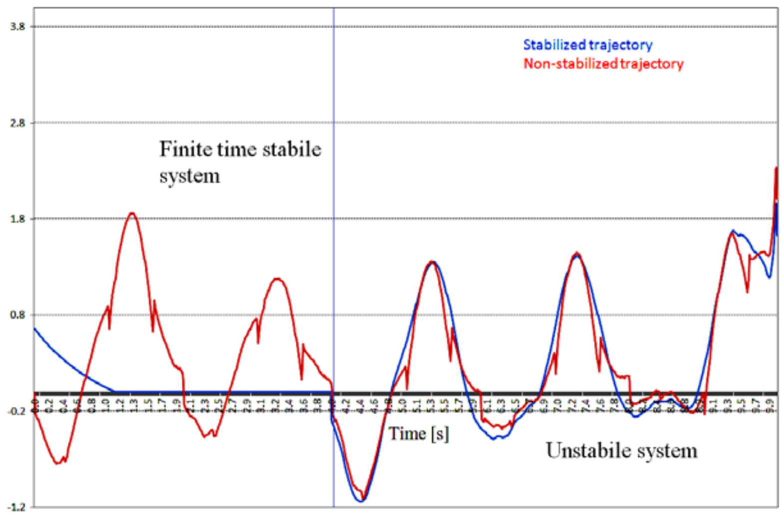


*Figure 5. The trajectories of the closed-loop stabilized system and non-stabilized system – a representative case corresponding to the ones in Fig.* 4 -

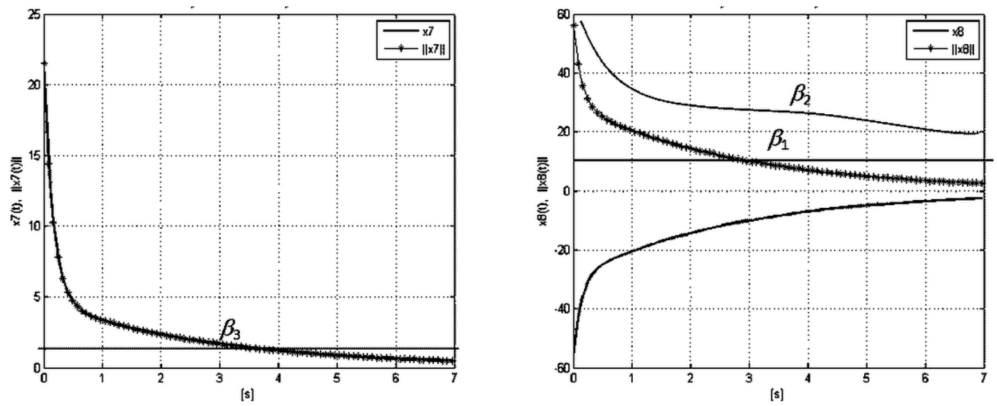


*Figure 6. Representative system trajectories and norms*

For $\beta_1$, the system is finite time unstable on $t\in[0, 3]$, since the condition (28) does not hold for chosen $t$.

If we choose new value $\beta_2$, system is finite time stable for any choice of $t$, no matter if delay is present or not.

Similarly, for some $\beta_3$, analyzed in the sense of *Theorem* 1, it was observed that the system is stable on $t\in[4, \infty[$.

## 6. CONCLUSION

Generally, this paper extends some of the basic results in the area of the non - Lyapunov stability to the particular class of LCSTDS. Furthermore, a part of this result is a geometric counterpart of the algebraic theory in [1] supplemented with appropriate criteria to cover the need for system stability in the presence of actual time delay terms. A novel sufficient delay-dependent criterion for the finite time stability, based on LMIs approach, has been established. The theory was validated and implemented on the robotic system for automatic drug delivery. The mathematical modeling, control and stability of the system were tested using the proposed approach.

## 7. ACKNOWLEDGMENT

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НЕЉАПУНОВСКА СТАБИЛНОСТ СИНГУЛАРНИХ СИСТЕМА: КЛАСИЧАН И МОДЕРАН ПРИЛАЗ СА ПРИМЕНОМ У АУТОМАТСКОЈ ИСПОРУЦИ ЛЕКОВА

**Сажетак:** У овом раду изведени су довољни услови практичне стабилности и стабилности на коначном временском интервалу за класу линеарних временски непрекидних сингуларних система са чистим временским кашњењем. Сингуларни системи и сингуларни системи са чистим временским кашњењем могу бити математички описани једначинама типа: $E\mathbf{x}'(t) = A\mathbf{x}(t)$ и $E\mathbf{x}'(t) = A_0\mathbf{x}(t) - A_1\mathbf{x}(t - \tau)$, следствено. Анализирајући стабилност на коначном временском интервалу изведени су нови услови, и то зависни и независни од временског кашњења.

Предложени прилаз се заснива на употреби Љапуновљевих функција и њиховим особинама на потпростору конзистентиних почетних функција или услова. Ове функције не морају бити позитивно одређене у целом простору стања, нити негатив-

но одређене дуж трајекторија система. Када се разматра практична стабилност, овај прилаз се комбинује са класичном љапуновском техником која гарантује особину привлачења система. У циљу добијања мање конзервативних резултата, коришћена је и ЛМИ метода. Предложени метод примењен је и тестиран на једном медицинском роботском систему. Систем је дизајниран за различите намене, као што су аутоматска испорука медикамената, биопсија или испорука радиоактивних зрнаца унутар оболелог ткива. За такав систем развијена је посебна техника моделирања, управљања и анализе стабилности описаног система. У сврху математичког моделирања, систем је декомпонован на механички део и на радну околину која пресудно утиче на динамичко понашање. Овакав приступ се показао адекватним у случају када спољашње силе утичу на динамику система. Добијен математички модел се анализира као сингуларни систем аутоматског управљања. У случају када се утицај спољашњих сила може занемарити, динамичко понашање се анализира класичним методама теорије управљања.

**Кључне речи:** системи са кашњењем, сингуларни системи, медицински робот.