Hybrid Methodology of Nonlinear Goal Programming

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Contribution to the state of the art

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Abstract: What we demonstrate here is a nonlinear goal-programming (NGP) algorithm based on hybrid connection of the modified simplex method of goal programming, gradient method of feasible directions and method of optimal displacement size finding-called HNGPM. Iterative methodology is given in five steps: (1) linearization the set of nonlinear constraints at particular point, (2) solving the problem of normalized linear goal programming, (3) feasible direction computation, (4) calculating optimal step length displacement, and (5) testing out convergence problem. Our idea was to apply Euler's theorem for the "total" linearization of the nonlinear constraints (in the space) around particular point. According to Euler's theorem, it is possible to apply this methodology to solve the problems of NGP whether the nonlinear constraint functions are linearly or positively homogeneous.

Keywords: Non-linear goal programming, Cobb- Douglas's production function, Euler's homogeneous function theorem, feasible directions method

Introduction

Here we use multi-objective optimization (also known as multi-objective mathematical programming) where the set of feasible solutions is not explicitly known in advance but it is restricted by constraint functions. Here we concentrate on nonlinear multi-objective optimization where at least one function in the problem formulation is nonlinear and ignore approaches designed only for multi-objective linear programming problems where all the functions are linear. In multi-objective optimization problems, it is characteristic that no unique solution exists but a set of mathematically equally good solutions can be identified. These solutions are known as non-dominated, efficient, non-inferior or Pareto optimal solutions.

Typically, in the Multicriteria Decision Making literature, the idea of solving a multi-objective optimization problem is understood as helping a human decision maker (DM) in considering the multiple objectives simultaneously and in finding a Pareto optimal solution that pleases him/her the most. Thus, the solution process needs some involvement of the DM in the form of specifying preference information and the final solution is determined by his/her preferences in one way or the other. In other words, a more or less explicit preference model is built from preference information and this model is exploited in order to find solutions that better fit the DM's preferences.[1]

Goal programming (GP) has proved to be an effective approach in facilitating decisions involving multiple objectives. Distinct from many other mathematical programming techniques, goal programming is able to overcome many limitations present in solving both single and multiple criteria problems.

The objective function of GP is minimization of positive and/or negative deviations from a set of goals determined by decision maker. Due to the fact that preventive priority and numerical differential weights are used, the difficulty of a priori estimation of a single objective is avoided. Besides, other multiple criteria decisions making requires approximation of weights for obtaining an objective function. However, the task of rank ordering some priority goals in goal programming is much easier for decision maker than assigning weights, because it approximates the actual decision making process.

Instead of determining an optimal solution in the manner of linear programming (LP), solutions based on GP satisfy ordinal priorities assigned to the goals. These solutions clearly point out: which goals can or cannot be achieved, the amount of underachievement connected with every unachieved goal, and the tradeoffs among the goals, so called Pareto optimum.

When comparing with the other management science techniques, GP could be characterized as a practical oriented application tool. It means direct help for decision makers, as well as effort for decreasing strict requires in ordering structural priorities in one form, which most of known multiple criteria methods of deciding are already familiar with.

In managing practice, we consider great number of problems as linear, but if we try to get to the base of problem, it will be discovered that they are nonlinear, and linearity is just an approximation. In problems of non-linear programming (NLP) the linearity could be subject either in constraints or in function criteria, or in both.

At LP the sphere of possible solutions characterizes convex set created on cut of the final number of linear constraints, so the possible problem solution is only the final number of extreme points. However, when at least one constraint is nonlinear equation, existence of no final number of extreme points is possible, or the solutions are located in no convex area, so there are many possible solutions of one problem. In many cases, there is no guarantee that the final solution generated by nonlinear goal programming (NGP) algorithm is the optimal solution unless certain conditions for the shape of the objective function and structural constraints are satisfied.

Nowadays, the great number of different methods for NLP solving problems is developed. One of the special group methods are numerical solving methods, in which the most important place takes methods of so called direct search, and then the gradient method.

In order to develop an algorithm for solving NLP models where Cobb-Douglas type nonlinear constraints exist, we have constructed one new gradient NLP algorithm with feasible directions methods built in, methodology named HNGP (Hybridized Non-linear Goal Programming). This methodology is based on linear GP algorithm and feasible directions methods. Using Euler's Homogeneous Function Theorem, HNGP makes linear each nonlinear homogeneous constraints function in the area of some particular point. [2]

METHODOLOGY CONSTITUENTS

The problem of nonlinear goal programming (NGP) can be expressed in the following way:

$$\text{Min } F(d) = \sum_{l=1}^{k}\sum_{i=1}^{n} w_i P_l (d_i^- + d_i^+) \qquad (1)$$

subject to:

$$G_i(x) = \sum_{j=1}^{n} g_j x_j + h_j \prod_{j=1}^{n} x_j^{e_j} + d_i^- - d_i^+ = c_i \qquad (2)$$

$$A_i(x) = \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (3)$$

$$x_j,\ d_i^-,\ d_i^+ \ge 0, \text{ for all } i=1,\dots,m,\ j=1,\dots,n. \qquad (4)$$

Where x_j are decision making variables, d_i^- and d_i^+ represent negative and positive deviation variables from the goals (underachievement and (overachievement), respectively. The g_{ij} are coefficients of linear portions of goals (constraints (2), a_{ij} are coefficients of structural constraints (3), h_{ij} are coefficients of nonlinear portions of goals (2), and e_{ij} are components. The c_i and b_i are constraints of right side in (2) and (3) respectively.

P_l in objective function (1) is preventive priority factors, so the following is valid:

$P_j >>> P_{j+1}$ for all j = 1, 2, . . . , k.

The highest priority is indicated by P_1, the next highest by P_2, etc. The w_i are weights assigned to some priority factors. The model of priorities means that P_1 is preferred than P_2 regardless of any weights w_i associated with P_2.

Euler's homogeneous function theorem

Homogeneous function is a function which has an attribute that for any real constant λ satisfies $F(\lambda x,\lambda y) = \lambda^n f(x,y)$, for a fixed n. Then we say that function F is homogenous by degree of homogeneity of n. If n>0, the function is positively homogeneous; if n=1, the function is linear homogeneous.

Example: The function

$z = f(x,y) = 3x^4+2x^2y^2+7y^4$

is homogeneous of degree 4 since

$f(\lambda x, \lambda y) = 3\lambda^4x^4 + 2\lambda^4x^2y^2 + 7\lambda^4y^4 = \lambda^4(3x^4+2x^2y^2+7y^4) = \lambda^4 \cdot f(x,y)$

If z=f(x,y) is positively homogeneous of degree n, and the first-order partial derivatives exist, then it can be shown that:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nf(x,y)$$

That is:

$$f(x,y) = \frac{1}{n}(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y})$$

This relation is known as Euler's homogeneous function theorem.

Example: According to Euler's theorem, for the function

$z=f(x,y)=3x^4+2x^2y^2+7y^4$

it is valid:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4f(x,y)$$

which can be verified as follows:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x(12x^3+4xy^2) + y(4x^2y+28y^3) = 12x^4+8x^2y^2+28y^4 = 4f(x,y)$$

In majority of up-to-date developed methods used for solving NGP problems, the most significant one of this very type is the gradient method combined with the feasible direction method [3].

The feasible directions method

The definition of a feasible direction reads: given a feasible point x, a direction vector d, is feasible if there exist s>0, so that x+sd is feasible

The feasible directions methods have been primarily intended for NGP and they are the iterative ones whose solutions in certain iterations have the following recursive form:

$x_{p+1} = x_p + l_p s_p(x)$ for all p=1,2,…,m

where $s_p(x)= s(x_0,x_1,x_2,\dots,x_p)$ is direction and $l_p \geq 0$ is a step size, which is chosen so that:

$F(x_p + l_p s_p) \leq F(x_p)$ for all $(x_p + l_p s_p) \in X$, and $0 \leq l_p \leq 1$, $s_p(x) \geq 0$,

And where:

$X = \{x \in R^n,$ and conditions (2), (3) and (4) are satisfied$\}$

If l_p is chosen in this way, it follows that all points between x_p and x_{p+1} are feasible.

Direct search NGP based methods utilize some type of logical search pattern or methods to obtain a solution that may or may not be the best satisfying solution. The logic process is based on repeated attempts to improve a given solution by evaluating its objective function and/or goal constraints.[4]

Gradient methods use the gradient direction as a direction for improving solution, thus defining the feasible and usable feasible directions, including reduction of the nonlinear problem to an approximate linear problem which is close to the initial one or some other by iterative set solution proceedings.

The method is iterative and each iteration starts with an initial feasible vector. In great number of iteration of the feasible direction method, feasible directions of improvement (usable feasible directions) are determined and a new "better" point is found out in that direction. Optimality is achieved when no further improvement can be made in any feasible direction.

To calculate the gradient, this method requires functions that are continuous and differentiable. In order to guarantee convergence of the algorithm the gradient method requires that the model constraints find a convex set at each goal level while the objective function is concave.

Many methods for solving linear or non-linear programming problems are developed based on feasible directions method. The only difference exist in extra requirements for fixing the initial point x_0, the directions $s_{k,}$ or the step lengths l_k.

The gradient of Cobb-Douglas's production function

In this case we are particularly interested in a nonlinear function of the Cobb-Douglas type, which general form is:

$$f(x_1, x_2, ..., x_n) = h\prod_{i=1}^{r} x_i^{b_i} \qquad (5)$$

where: $x_i \geq 0$, for all i = 1,2,..., n, are independent variables, $h \in R$ is coefficient, and $b_i \in R$ are exponents of independent variables. This is most similar and most frequently applied form of production function by which the production is rated in a certain economy, expressed as a function of labor and money investment.

In accordance with Euler's homogeneous function theorem conditions, it has to be noted if $f(x_1, x_2, ..., x_n)$ is homogenous of degree r and the initial derivatives exist, than it can be shown that is:

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{r} \sum_{i=1}^{n} \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i}\bigg[_{x_p}\bigg] \cdot x_i$$

where is with $[_{x_p}]$ denoted "in particular point x_p", and where for Cobb-Douglas's functions (5):

$$\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_k}\bigg[_{x_p}\bigg] = hb_k x_k^{(b_k-1)} (\prod_{\substack{i=1 \\ i\neq k}}^{n} x_i^{b_i})$$

for all $x_i \geq 0$ (6)

are the coefficients of linear constraints calculated in particular point x_p (denoted as).

HNGP methodology

The main idea of HNGPM is that no linearity of constraints set, using Euler's homogeneous function theorem, linearize in the neighbor of point x_p (float feasible solution). Then, using modified simplex linear goal programming method, one could find a feasible direction L_p. Subsequently, using one direct displacement along the feasible direction step length searching procedure, we find out optimal step length s_p in that feasible direction such is the approximation valuation made by linearity in this new point and in satisfied limits of accuracy. This procedure is iterative and repeating until the shift made in one of the next iterations would be less than of convergence criterion

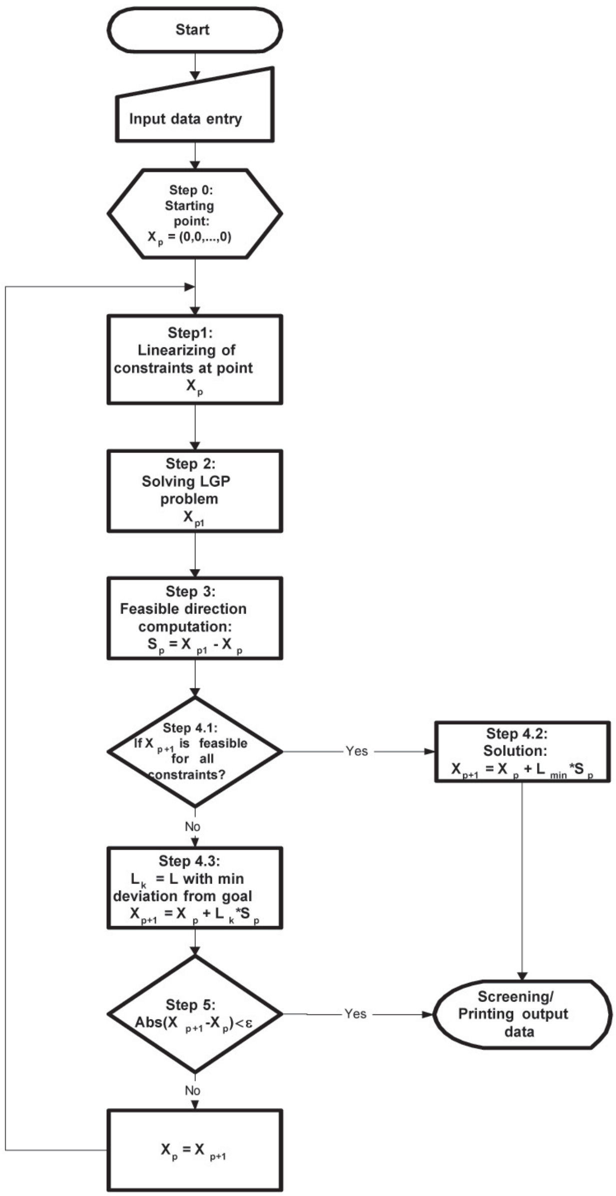
Algorithm of HNGPM

Algorithm of HNGP is based on hybrid connection of modified simplex method of GP, gradient method of feasible directions, and method of optimal displacement size finding. Concerning this problem, the objective function is linear and only a few constraints are nonlinear, the procedure is simplified when compared with general convex programming problems.

Iterative proceedings are given in six steps with an initial step (Step 0) used only at the beginning, i.e. in the initial iteration (Picture 1).

The initial step sets all vector solution values to zero and uses this point as origin x_0. In this point all nonlinear portions gradient values of the goals are equal to zero. This reduces the problem of NGP (problem of (1) to (4)) to linear approximation solvable by the modified simplex method of linear goal programming.[3]

The first step of an HNGPM algorithm is the computation of gradients of all constraints together with checking the status of nonlinear constraints in order to (if need) deflect, the gradient of active (binding) nonlinear constraints.; to avoid zigzagging, which is often possible in solving problems algorithm of NLP. Without this deflecting, the algorithm may converge upon a sub optimal point.



In the second algorithm step we solve normalized NGP problem of (1) to (4), that is:

$$\text{Min } F(d) = \sum_{l=1}^{k}\sum_{i=1}^{n} w_i P_l (d_i^- + d_i^+) \qquad (7)$$

subject to:

$$G_i(x) = \sum_{j=1}^{n} \nabla g_{ij}(x_p) x_j + d_i^- - d_i^+ = nc_i \qquad (8)$$

$$A_i(x) = \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (9)$$

$$x_j,\ d_i^-,\ d_i^+ \ge 0,\ d_i^- \cdot d_i^+ = 0, \text{ for all } i=1,\ldots,m,\ j=1,\ldots,n. \qquad (10)$$

and where sign $\nabla g(x_p)$ denotes the gradient of non-linear constraint function in particular point x_p calculated by (6). That is, nonlinear constraints are transformed to linear on the basis of Euler's theorem computed gradient value in point x_p (in the initial step it was x_0).

In this step linear goal-programming specified problem (7) to (10) is solved by modified simplex method of linear goal programming. This method of linear goal programming derives the feasible solution x_p.

The third algorithm step serves for feasible direction $d_p(x)$ computation:

$$d_p(x) = x_p' - x_p$$

which has to be "searched" according to the constraints and structure of priority in order to improve the simplex solution.

In the fourth step, the optimal displacement step size of the solution vector is determined by linear searching along with feasible direction identified in the previous step. In this way, first the structural constraints $A_i(x)$ must be satisfied and, after that, the goal ones, starting from the goal constraint $G_i(x)$ from (2) which contains the deviation variable with top priority (P_1) in the objective function of the NGP problem, minimum deviation in the function criterion, and then other priorities in lexicography importance order (P_2, P_3, itd.).

To perform the search and to prevent infinite moving in the cases of unlimited problems, it is necessary to determine the lower and upper displacement limits in feasible direction and unit displacement size (increment) within these limits.

The convergence of this very problem, speed of convergence, and other algorithm features depend on the choice of vector $s_p(x)$ and limits for the step size s_p. In initial step the lower limit of s_p in the algorithm was set to 0 and the upper to 1. If it is necessary it is possible to set them differently. The unit displacement size in the algorithm was set on 0.1 at first, but it is possible to increase the search density.

Using these procedures, we define whether the first goal constraint (per priority and not per order) is satisfied within the initial limits. If it is satisfied at some short interval [s_{pl}, s_{pd}], for all $s_{pl} \geq 0$ and $s_{pd} \leq 1$, the search for the further goal constraint goes on exclusively within this interval as its satisfaction, in accordance with Pareto optimality, cannot be sought to the detriment of satisfaction of a higher priority goal.

If we during the algorithm search come across goal constraint that could not be satisfied in limits of feasible directions set for previous higher priority, then within these limits the algorithm identifies the value of displacement size for which the deviation from the subject constraint is the least s_p ($0 < s_p \leq 1$), we complete the searching and compute the new solution (successor) as follows:

$$x_{p+1}=x_p+s_pL_p(x_p)$$

If all constraints are satisfied, then the new solution is at the same time the optimal solution to this very problem.

In the fifth algorithm step we check the problem convergence by previously set small value of the desired level of convergence accuracy -- ε:

$$|(x_{p+1} - x_p| \leq \varepsilon, \text{ for all p = 1,2,...etc.}$$

which means that the problem has converged and the algorithm finalized its work. Otherwise, the optimum search procedure continues beginning with the first algorithm step.

CONCLUSIONS

The idea of linearization of nonlinear constraint and solution of nonlinear programming problems is not new, because Griffith and Stewart (in 1961) first suggested that nonlinear problem may be linearized around the particular point by expansion as a Taylor's series, ignoring elements of a higher order than linear and adding two more restrictions for each nonlinear constraints. In that way, nonlinear programming problems have been transformed into a form which can be solved by the linear programming methods. However, there are other optimal methods which can solve programming problems in which nonlinear constraints are not Cobb-Douglas type. Yet, among them are very small number of nonlinear multi-criterial programming methods, especially goal programming.

Our idea of HNGP methodology was to apply Euler's theorem for the "total" linearization of nonlinear constraints around the particular point. This methodology was possible using this very theorem, as well as its utilization for solving nonlinear goal programming problems. In economic theory, the production function is frequently assumed to be linearly homogenous, because such functions have convenient characteristics. Although HNGPM is a numerical methodology for solving only certain types of NGP problems, it could be extended to solve other nonlinearly constraints forms.

In general, HNGPM methodology is used for solving nonlinear goal programming problems in which constraints are given by other homogeneous, continuous and differentiable functions. The only request the algorithm should obey is to do special subprograms for computing of nonlinear function gradient, and computing vector solution.

The advantage of our approach is unnecessarily adding two new constraints in every simplex iteration algorithm for every linearization of nonlinear constraint. Another advantage is that linearization of nonlinear constraints is more precise, with which algorithm along with other necessary conditions of convergence the problem solution is found more rapidly.

This work shows that with using Euler's theorem it is possible to perform hybrid connection of feasible directions gradient method with linear goal programming method, and create brand new methodology. These resultants are opening new possibilities for further hybridization of nonlinear goal programming method, and creating new and more effective methodologies options, as well as setting of interactive programming.

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